Formulation of the STT Track Fit
DØ Note xxxx

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Abstract

The trajectory of a charged particle in the plane transverse to a magnetic field is a circle in the absence of multiple scattering. For particles which originate near the origin, the circular path can be written as

$$\phi(r) = \frac{b}{r} + \kappa r + \phi_0$$  \hspace{1cm} (1)

in which $\phi$ is the azimuthal angle of the point on the track at radius $r$, $b$ is the distance-of-closest-approach of the track to the origin, $\phi_0$ is the angle of the track at the point-of-closest-approach and $\kappa \equiv 1/(2R)$ is the track curvature. Both $b$ and $\kappa$ are signed. This form represents an approximation to the circle under the condition $\kappa r \ll 1.0$, $br \ll 1.0$ and $\phi - \phi_0 \ll 1.0$. For DØ this approximation holds for $|b| < 2$ mm and for all relevant transverse momenta.

This form can be used in a $\chi^2$ minimization to find the parameters of a given charged particle track. The $\chi^2$ is given in general by

$$\chi^2 = \sum_{hit} \left[ \frac{(\phi(r_i) - \phi_i)}{\sigma_{\phi_i}} \right]^2$$

in which $(r_i, \phi_i)$ are reconstructed hits in the DØ silicon and fiber tracking detectors and $\sigma_{\phi_i}$ is the error in $\phi$. A more convenient form involves not the angular hit resolution, but the resolution $\sigma_x$ measured in the plane approximately perpendicular to the track trajectory. In this form, $\sigma_{\phi_i} \equiv \sigma_x/r_i$. This form is useful because one needs only two resolution values $\sigma_{CFT}$ and $\sigma_{SMT}$ for hits from the fiber and silicon detectors respectively.\footnote{For the most precise results, hit-by-hit resolutions should be used. However, given the initial approximation, the method used here holds.} In this case the $\chi^2$ becomes (after substituting eqn. 1)

$$\chi^2 = \sum_{hit} \left[ \frac{b + \kappa r_i^2 + \phi_0 r_i - \phi_i r_i}{\sigma_x i} \right]^2$$  \hspace{1cm} (2)
The best fit to the track parameters is found by minimizing this $\chi^2$ with respect to $b$, $\kappa$ and $\phi_0$ in the usual manner. Because $\phi(r)$ is linear in the parameters, this can be performed analytically. Taking the derivative of eqn. 2 with respect to each track parameter gives:

$$\frac{1}{2} \frac{\partial \chi^2}{\partial b} = \sum_{hi} \left( \frac{b + \kappa r_i^2 + \phi_0 r_i - \phi_i r_i}{\sigma_i^2} \right)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial \phi_0} = \sum_{hi} r_i \left( \frac{b + \kappa r_i^2 + \phi_0 r_i - \phi_i r_i}{\sigma_i^2} \right)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial \kappa} = \sum_{hi} r_i^2 \left( \frac{b + \kappa r_i^2 + \phi_0 r_i - \phi_i r_i}{\sigma_i^2} \right)$$

If we define $c_{ij} \equiv r_i^2/\sigma_i^2$ and $\Phi_{ij} \equiv \phi_i r_i^2/\sigma_i^2$, $c_j \equiv \sum_i c_{ij}$ and $\Phi_j \equiv \sum_i \Phi_{ij}$ and rearrange terms, the above equations reduce to the linear system

$$\Phi = Cp$$

with

$$p = \begin{pmatrix} b \\ \phi_0 \\ \kappa \end{pmatrix}, \quad C \equiv \begin{pmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix} \quad \text{and} \quad \Phi \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}$$

This can be solved in the usual manner, giving $p = C^{-1}\Phi$. For completeness, the inverse is

$$C^{-1} = \frac{1}{D} \begin{pmatrix} c_4 c_2 - c_3^2 & c_2 c_3 - c_1 c_4 & c_1 c_3 - c_2^2 \\ c_2 c_3 - c_1 c_4 & c_0 c_4 - c_2^2 & c_1 c_2 - c_0 c_3 \\ c_1 c_3 - c_2^2 & c_1 c_2 - c_0 c_3 & c_0 c_2 - c_1^2 \end{pmatrix}$$

with the determinant given by $D = c_0 c_2 c_4 + 2c_1 c_2 c_3 - c_0^2 c_3 - c_2^3 - c_1^2 c_4$. When computing the inverse only the upper triangular terms are computed, and the lower half is obtained by the symmetry condition. The resulting track parameters are given by

$$b = \frac{1}{D} \left[ (c_4 c_2 - c_3^2) \Phi_1 + (c_2 c_3 - c_1 c_4) \Phi_2 + (c_1 c_3 - c_2^2) \Phi_3 \right], \quad (3)$$

$$\phi_0 = \frac{1}{D} \left[ (c_2 c_3 - c_1 c_4) \Phi_1 + (c_0 c_4 - c_2^2) \Phi_2 + (c_1 c_2 - c_0 c_3) \Phi_3 \right], \quad (4)$$

$$\kappa = \frac{1}{D} \left[ (c_1 c_3 - c_2^2) \Phi_1 + (c_1 c_2 - c_0 c_3) \Phi_2 + (c_0 c_2 - c_1^2) \Phi_3 \right]. \quad (5)$$

Because $\phi(r)$ is linear the track parameters, the parameter error matrix

$$\epsilon \equiv C^{-1}, \quad (6)$$

and the $\chi^2$ is found by substituting the parameters back into eqn. 2 and rearranging to get

$$\chi^2 = \sum_{i=hi} c_{i0} [b + (\kappa r_i + \phi_0 - \phi_i) r_i]^2 \quad (7)$$

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1 Alternate Format

The above method for writing the linear equation is the most compact. However, it does not have the best numerical properties. The elements of the $\Phi$ vector have different dimensions which leads to very different numerical scales for each of the terms in both $\Phi$ and the inverse matrix. Because the $\Phi$ vector is linear in $\phi_i$, the problem can be rewritten with the vector $\Phi' \equiv (\phi_1, \phi_2, ..., \phi_N)$ with $N$ the number of hits. The matrix is then a (nonsymmetric) $3xN$ matrix. Let the original inverse matrix be defined as $M$ with elements $M_{ij}, \ i = 1, 3, \ j = 1, 3$. In terms of this, the elements of the new matrix are

$$M'_{jk} = M_{j1} r_k / \sigma_k^2 + M_{j2} r_k^2 / \sigma_k^2 + M_{j3} r_k^3 / \sigma_k^2$$

with $k = 1, N$ and $N$ the number of hits used in the fit. The track parameters are found from $p = M' \Phi'$ which is simply

$$b = \Sigma_k M'_{1k} \phi_k$$

$$\phi_0 = \Sigma_k M'_{2k} \phi_k$$

$$\gamma = \Sigma_k M'_{3k} \phi_k$$

with $k = 1, N$ and $N$ as above. The operation count increases by $3x(N - 3)$ over the initial $3x3$ form, but all terms in a given row of the matrix and all terms in the vector have the same scale. This significantly reduces the demand for high precision arithmetic. Further, a track-by-track rotation can be performed, such that at least one of the $\phi_k = 0$, reducing the number of operations.

Figure 1 shows the difference in impact parameter from a fit performed in a reduced precision floating point format and a fit using standard double precision floating point. If a floating point calculation is performed in programmable logic, reduced precision will be necessary to obtain sufficient speed. One sees clearly the improved numerical stability of the $3xN$ formulation. Such an improvement may also permit the calculation to be performed using integers, provided the matrix is precomputed and stored in a look up table.

Appendix A: Linearized fit in the $rz$ Plane

In the linear approximation above, the $r\phi$ and $rz$ parameters decouple, giving the $rz$ form of the track as

$$z(r) = r \times \tan \lambda + z_0$$

with $\tan \lambda$ and $z_0$ the track parameters. The parameter $\tan \lambda$ is the dip angle, and $z_0$ is the $z$ coordinate of the track at the point-of-closest approach in the $r\phi$ plane. The $\chi^2$ for the $rz$ plane fit is given by

$$\chi^2 = \sum_{hits} \left[ \frac{z_0 + r_i \tan \lambda - z_i}{\sigma_{zi}} \right]^2$$

Minimizing this $\chi^2$ analytically, and defining $d_j^i \equiv \sum_i r_i^j / \sigma_{zi}^2$ and $Z_j \equiv \sum_i z_i r_i^j / \sigma_{zi}^2$ gives

$$Z = C'p'$$
with

\[ p' \equiv \begin{pmatrix} z_0 \\ \tan \lambda \end{pmatrix}, \quad C' \equiv \begin{pmatrix} c'_2 & c'_1 \\ c'_1 & c'_0 \end{pmatrix} \quad \text{and} \quad Z \equiv \begin{pmatrix} Z_1 \\ Z_0 \end{pmatrix} \]

This can be solved in the usual manner, giving \( p' = (C')^{-1}Z \). The inverse is

\[
(C')^{-1} = \frac{1}{D^*} \begin{pmatrix} c'_0 & -c'_1 \\ -c'_1 & c'_2 \end{pmatrix}
\]

with the determinant given by \( D' = c'_0 c'_2 - c'_1 c'_1 \). The resulting track parameters are given by

\[
\begin{align*}
  z_0 &= \frac{1}{D'} \left[ c'_0 Z_1 - c'_1 Z_0 \right], \\
  \tan \lambda &= \frac{1}{D'} \left[ c'_1 Z_1 - c'_2 Z_0 \right].
\end{align*}
\]

Because this is linear in the track parameters, the error matrix is given by

\[ \mathcal{E}' \equiv (C')^{-1} \]

**History**

- Version 2
  
  1. Added error matrix comment to \( r \phi \) fitting
  2. Added \( rz \) appendix for reference
Figure 1: Two distributions of the difference in impact parameters resulting from a reduced precision fit and a standard double-precision fit. The left-hand panel is for a fit performed using the $3xN$ form of the linear algebra; the right-hand side is for the $3x3$ form. In both cases, the reduced precision floating point format has an 18-bit mantissa and a six bit exponent, and the matrix $M$ was precomputed using full double precision. The superior numerical stability of the $3xN$ form of the problem is clear.