1. Show in detail the symmetry principle responsible for, and why and how it leads to, the conservation of electrical charge.

**Solution:**

The symmetry that is invoked is the local phase symmetry (local gauge symmetry) of the U(1) type, i.e. invariance of the Lagrangian under transformation of the fields (operators) $\phi \rightarrow \phi' = e^{i\alpha(x)}\phi$, where $\alpha(x)$ is an arbitrary scalar function of space-time coordinate $x$. Requiring such a symmetry of the Lagrangian forces the introduction of a gauge vector boson field $A^\mu$ into the Lagrangian in a very specific way, namely such that its introduction absorbs the extra terms that are resulting from the action of the four-derivative operator $\partial^\mu$ on the $x$-dependent exponent function $\alpha(x)$: we are forced to replace $\partial^\mu$ everywhere by the ‘generalized derivative’ $D^\mu = \partial^\mu + i q A^\mu$. The required simultaneous transformation of $A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \alpha(q)$ cancels with the extra terms $\partial^\mu \alpha$ while it leaves the observable fields $E$ and $B$ unchanged. The introduction of the photon field this way, allows us to also introduce a ‘kinetic term’ for the free photon using the anti-symmetric tensor $F^{\mu\nu}$, which respects the gauge invariance. Finally, one can derive a conserved current.

2. Examine the processes listed, and state for each one whether it is possible or impossible, according to the Standard Model (which does not include GUTs, with their potential violation of the conservation of lepton number and baryon number). In the former case, state which interaction is responsible; in the latter case cite a conservation law that prevents it from occurring. (Following custom, I will not indicate the charge when it is unambiguous, thus $\gamma$, $\Lambda$, $n$, and $\eta$ are neutral; $p$ is positive.)

<table>
<thead>
<tr>
<th>Process</th>
<th>Solution</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\pi^- + p \rightarrow p + K^-$ ($\sqrt{s} &gt; m_p + m_K$)</td>
<td>Strangeness not conserved</td>
<td>(b) $A^\mu \rightarrow p + \gamma$</td>
</tr>
<tr>
<td>(c) $\eta \rightarrow \pi^0 + e^+ + e^-$</td>
<td>$C$ not conserved: $C(e^+ e^-) = C(\gamma) = -1$; $C(\pi) = C(\eta) = -1$</td>
<td>(d) $\eta \rightarrow \pi^0 + \gamma + \gamma$</td>
</tr>
<tr>
<td>(e) $D^0 \rightarrow K^0 + \pi^0$</td>
<td>Allowed; via the $K^0\pi^0$ final state...</td>
<td>(f) $D^0 \rightarrow K^+ + \pi^-$</td>
</tr>
<tr>
<td>(g) $\nu_e + p \rightarrow e^+ + \Lambda + K^0$</td>
<td>Not allowed: Lepton number violation</td>
<td>(h) $\bar{\nu_e} + p \rightarrow n + e^+$ ($\sqrt{s} &gt; m_n + m_e$)</td>
</tr>
<tr>
<td>(i) $\Sigma^+ \rightarrow n + e^+ + \nu_e$</td>
<td>Not allowed: only possible weak decay $uuu \rightarrow uuu + W^-$ (so-called $\Delta S = \Delta Q$ rule)</td>
<td>(j) $\Omega^- \rightarrow \Lambda + \pi^-$</td>
</tr>
</tbody>
</table>
3. Describe in less than 200 words the crucial experimental details and experimental findings that showed unequivocally that Parity is violated in the weak interaction.

Solution:
One must start with a well defined spin state, and end with a well defined spin state, in order to deduce the spin of the electron and neutrino emitted in the weak decay. Hence, C.S.Wu et al. took nuclear spin-polarized Cobalt 60 (needs close to absolute zero cooling to overcome thermal randomization). The observation was that electrons from the (weak) beta decay are emitted only in directions opposite to the spin they carry, i.e. only left-handed electrons and righthanded antineutrinos are emitted. Cross checks are made by letting the sample warm up (electron emission now becomes symmetric), and by reversing the polarizing magnetic field (the electron angular distribution flips also).

4. Show that the phase-space element $\frac{dp}{E}$ is a Lorentz invariant.

Solution:
$$\frac{dp'}{E'} = \frac{d(\gamma p - \beta \gamma E)}{\gamma E - \beta \gamma p} = \frac{\gamma dp - \beta \gamma dE}{\gamma E - \beta \gamma p} = \frac{\gamma dp - \beta \gamma 2p \cdot dp/(2E)}{\gamma E - \beta \gamma p} = \frac{(\gamma E - \beta \gamma p)dp}{(\gamma E - \beta \gamma p)E} = \frac{dp}{E}$$

5. Show that $(\sigma \cdot p)^2 = |p|^2 1$.

Solution:
$$(\sigma \cdot p)^2 = \sigma_i \sigma_j p_i p_j = \frac{1}{2}(\sigma_i \sigma_j + \sigma_j \sigma_i)p_i p_j + \frac{1}{2}(\sigma_i \sigma_j - \sigma_j \sigma_i)p_i p_j = \delta_{ij} p_i p_j + i \epsilon_{ijk} \sigma_k p_i p_j = p_i^2 + p_j^2 + i \alpha (p \times p)_k = (p_x^2 + p_y^2 + p_z^2) \mathbf{1}_2 + 0 = |p|^2 \mathbf{1}_2$$

6. Starting with the Dirac Lagrangian $L_D = \bar{\psi}(x)\left[i \gamma^\mu \partial_\mu - m\right]\psi(x)$, where $\bar{\psi} = \psi^\dagger \gamma^0$, derive the Dirac equations for $\psi$ and $\bar{\psi}$ as the Euler-Lagrange equations of $L_D$.

Note: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ (Dirac representation); $\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$, $\sigma^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$0 = \partial_\mu \frac{\partial L}{\partial (\partial_\mu \psi)} - \frac{\partial L}{\partial \psi} = i(\partial_\mu \bar{\psi})\gamma^\mu + m\psi = 0$$, and
$$0 = \frac{\partial L}{\partial \bar{\psi}} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \bar{\psi})} = (i\gamma^\mu \partial_\mu - m)\bar{\psi} = 0$$

7. Show, by explicit substitution, that a solution of the Dirac equation, $\psi(x) = u(p) e^{-i px}$, is also a solution of the Klein-Gordon equation.

Solution:
The only $t$ and $x$ dependence is in the exponential function.
$$\partial^2 \psi(x,t)/\partial t^2 - u(p) e^{-i px} (-iE)^2 =?= (\nabla^2 - m^2) \psi(x,t) = u(p) e^{-i px} \left[ (i p)^2 - m^2 \right]$$
$$-E^2 =?= [-p^2 - m^2]$$, which indeed is the relativistic energy-momentum-mass relationship.
8. Show that $\left\{ \gamma^5, \gamma^\mu \right\} = 0$, and that $\bar{\psi} \gamma^5 \gamma^\mu \psi$ behaves like a pseudovector, i.e. as a vector under proper Lorentz transformations and as a pseudovector under parity.

9. Give an expression for the fermion vertex factor in case of a weak interaction that only couples to left-handed particles (and right-handed antiparticles).

10. Using the Feynman rules for QED and the weak interaction, draw and label(!) ALL the diagrams and construct the accompanying matrix elements $M_{fi}$ for the following processes. What do you propose to do with quark lines that are contained within the same hadron?

Note 1: just write down the full matrix elements, do not do any calculations…

Note 2: draw the quark-based diagrams for the hadrons (except when explicitly called “point-like”).

(a) $e^+ + e^- \rightarrow e^+ + e^-$
(b) $\pi^+ + e^- \rightarrow \pi^+ + e^-$ (point-like pions!)

(c) $\pi^+ + \pi^- \rightarrow e^+ + e^-$ (point-like pions!)
(d) $\gamma + e^- \rightarrow \gamma + e^-$

(e) $e^+ + e^- \rightarrow \nu_e + \bar{\nu}_e$
(f) $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$

(g) $A \rightarrow n + \pi^0$
(h) $K^+ \rightarrow \pi^+ + \pi^0$

(i) $A_c^+ \rightarrow p + \bar{K}^0(892)$; $A_c$ is a charmed $C=+1$ baryon.
(j) $\eta \rightarrow \gamma + \gamma$