I Short answer problems (110 points)

a. QCD: list at least two pieces of experimental evidence for the existence of the color quantum number. (6 pnts)

i) The existence of the fully flavor-spin-orbital symmetric states $\Delta^0$, $\Delta^-$, or $\Omega^-$, which nevertheless describe fermions. Antisymmetry under color provides the necessary overall antisymmetry. ii) Another piece of evidence comes from the pi-zero lifetime (which would be incorrectly predicted by a factor three without color), and iii) from the measurement of electron-positron annihilation into muons versus annihilation into quarks (i.e. hadrons): this ratio would be three times larger if color did not exist.

b. Examine the following processes, and state for each one whether it is possible or impossible, according to the Standard Model (which does not include GUTs, with their potential violation of the conservation of lepton number and baryon number). In the former case, state which interaction is responsible; in the latter case cite a conservation law that prevents it from occurring. (Following the usual custom, I will not indicate the charge when it is unambiguous, thus $\gamma$, $\Lambda$, and $n$ are neutral; $p$ is positive.) (32 pnts)

(a) $p + \bar{p} \to \pi^+ + \pi^- + \pi^0$ S (annihilation)  
(b) $\pi^0 \to \gamma^+ + \gamma$ EM (q$+q\to\gamma^+\gamma$)  
(c) $\pi^+ \to \pi^0 + e^+ + \nu_e$ W (small Q value)  
(d) $\Sigma^- \to n + \pi^-$ W ($s\to u+W^\pm$)  
(e) $\Delta^{++} \to n + \pi^+ + \pi^+$ X (energy cons.)  
(f) $\bar{\nu}_e + p \to n + e^+$ W (if $\sqrt{s}>m_p$)  
(g) $D^0 \to K^- + \pi^+$ W ($c\to s+W^\pm$)  
(h) $p + p \to \Sigma^+ + n + K^0 + \pi^+ + \pi^0$ S (if $\sqrt{s}>m_t$)  
(i) $\rho^0 \to \pi^0 + \pi^0$ X (B-E stats)  
(j) $\bar{p} + p \to p + p + p + \bar{p}$ S (if $\sqrt{s}>4m_p$)  
(k) $D^+ \to K^- + \pi^+ + \pi^+$ X ($c \to \neq s$)  
(l) $\pi^+ + n \to \pi^0 + p$ S  
(m) $\Sigma^0 \to \Lambda + \gamma$ EM ($\nu_s\to\nu_s$: magn.)  
(n) $\Xi^- \to \Lambda + \pi^-$ W ($s\to u+W^\pm$)  
(o) $\pi^- + p \to \Lambda + K^0$ X (strangeness)  
(p) $\Sigma^- \to n + e^+ + \bar{\nu}_e$ W ($s\to u+W^\pm$)

c. Starting from a fully-stretched state $u\bar{d}$ : i) show that the $\pi^0$ (or $\rho^0$) will be a state composed of $
\frac{1}{\sqrt{2}}(|d\bar{d}) + |u\bar{u})\rangle = \frac{1}{\sqrt{2}}(|d\bar{d}) - |u\bar{u})\rangle$. ii) Show that the $\omega = \frac{1}{\sqrt{2}}(|d\bar{d}) + |u\bar{u})\rangle$ is orthogonal to $\rho^0$ and an iso-singlet. iii) Using the observed decays of the $\phi(1020)$ meson, argue that its quark content is $ss$. (12 pnts)

i) Taking $\Gamma \rho^+ = \Gamma |u\bar{d}\rangle = \sqrt{[(3/4) - (1/4)]} \langle |d\bar{d}) + |u(-\bar{u})\rangle = \sqrt{1/2} \{ |d\bar{d}) + |u(-\bar{u})\rangle \} = \rho^0$. ii) The orthonormal state is the singlet $\omega = \sqrt{1/2} \{ |d\bar{d}) - |u(-\bar{u})\rangle \}$; using the orthonormality of the $|d\bar{d})$ and $|u\bar{u})$ states one finds $\langle \omega|\rho^0\rangle = 1 - 1 = 0$. iii) The dominant decay modes of the $\phi(1020)$ meson are into Kaons, even though the Q-value of those decays is minute, about 40 MeV. That indicates that the parent meson contained the strange quarks that appear in the decay Kaons. In this picture, $\rho\pi$ and $3\pi$ decay modes originate from the annihilation of the $ss$ quarks in the $\phi$ into gluon(s).
d. Consider the neutral vector mesons $\rho$, $\omega$, $\phi$, and $J/\psi$, which all have $J^P = 1^{--}$. Assuming that their (electromagnetic!) decay into an electron-positron pair proceeds via $\bar{q}q'$ annihilation into an intermediate (virtual) photon, show that their decay widths into $e^+e^-$ are in the ratio $\Gamma_{\rho} : \Gamma_{\omega} : \Gamma_{\phi} : \Gamma_{J/\psi} = 9:1:2:8$. Neglect possible effects from their difference in mass, and purely look at their quark content. (12 pnts)

The decay amplitude has a coupling constant between quark and photon proportional to the constituent quark charge. The $\rho = 1/2\sqrt{1/2} \{|u \bar{d}\rangle + |d \bar{u}\rangle\}$, i.e. the proportionality factor in the decay amplitude is $1/2(1/3 + 2/3)^2 = 1/2$; for the $\omega = 1/2\sqrt{1/2} \{|u \bar{d}\rangle - |d \bar{u}\rangle\}$: $1/2(1/3 - 2/3)^2 = 1/18$; for the $\phi = ss$: $(1/3)^2 = 1/9$; and for the $J/\psi \approx cc$: $(2/3)^2 = 4/9$.

e. Describe the evidence found by C.S. Wu et al. for parity violation in the weak interaction:
   i) Describe the particle physics process that was studied, spins, etc.
   ii) Describe the experimental setup; but only the crucial points!
   iii) Describe the experimental outcome, and why it supports violation of parity (describe what parity conservation would have predicted). (12 pnts)

See lecture notes.

f. The $K^0$ produced in strong interactions is (to an excellent approximation) a superposition of two states with opposite CP parity: the short-lived $K_1^0$ and long-lived $K_2^0$. (21 pnts)
   i) List a decay mode of each CP eigenstate that proves its CP eigenvalue.
   ii) Argue why these the CP states have very different lifetimes.
   iii) Do the CP eigenstates have well-defined strangeness? Argue!
   iv) Describe the experimental evidence for CP violation in the $K^0$ system.
   v) Explain the phenomenon of $K_1^0$ regeneration.

See lecture notes

g. Show that the Mandelstam variable $t = (p_1 - p_3)^2$, for the case that $m_1 = m_3$ and $m_2 = m_4$, can be written in the CM system exactly as $2p^* \cdot (1 - \cos \theta^*)$, where $|p^*|$ stands for the length of the three-momentum vector in the CMS, and $\theta^*$ is the angle between $p_1^*$ and $p_3^*$. (8 pnts)

\[ t = (p_1 - p_3)^2 = m_1^2 - 2E_1E_3 + 2|p_1||p_3|\cos \theta_{13}. \]
In the CMS and for $m_1 = m_3 = m$: $E_1 = E_3 = E$ and $|p_1| = |p_3| = |p|$, thus: $t = 2m^2 - 2E^2 + 2|p|^2 \cos \theta^* = -2|p|^2 + 2|p|^2 \cos \theta^* = -2|p|^2 (1 - \cos \theta^*).$

h. Show that the phase-space element $d\mathbf{p}/E$ is a Lorentz invariant. (7 pnts)

\[ d\mathbf{p}'/E' = d(-\beta \gamma E + \gamma \mathbf{p})/(\gamma E - \beta \gamma) = (-\beta \gamma dE + \gamma dp)/(\gamma E - \beta \gamma) = (-\beta \gamma dp/E + \gamma dp)/(\gamma E - \beta \gamma) = d\mathbf{p}/E \]
II Scattering of spinless particles: (90 points) 

\[ \frac{d\sigma}{d\Omega^\ast} \] and the total cross section \( \sigma_{\text{tot}} \) for a weak interaction in lowest-order of spin-0 Klein-Gordon bosons \( e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e \) at CM energy of \( \sqrt{s} = \sqrt{10} \text{ GeV} \). Assume the “weak charge” of the spinless electron and neutrino is \( g_w \), with \( \alpha_w \equiv g_w^2/(4\pi) \approx 1/29 \). Ignore particle masses at this energy!

a. Argue why the ingoing (i) and outgoing (f) particles that participate in an interaction, can justly be described by solutions of the free-particle equation of motion. (6 pts)

The in- and outgoing particles are defined and measured far away (in both space and time) from the interaction region and interaction moment.

b. Derive the expression for the K-G current \( j_{\mu}^i(x) \) from the Klein-Gordon equation of motion:

\[ (\partial_{\mu} \partial_{\mu} + m^2) \psi(x) = 0. \]

Express the current in terms of the \( i \) and \( f \) fourmomenta \( p_i \) and \( p_f \). (9 pts)

The K-G current can be derived by taking the K-G equation, multiplied from the left by \( \psi_i^* \), and subtract the complex conjugate of the K-G equation, multiplied from the right by \( \psi_i \):

\[ \psi_f^*(\partial^2 + m^2)\psi_i = 0 \Rightarrow \psi_i^*\partial^2 \psi_i - \psi_i \partial^2 \psi_i^* = \partial_{\mu}(\psi_i^*\partial^\mu \psi_i - \psi_i \partial^\mu \psi_i^*) = 0; \quad j_{\mu}^i = ie(\psi_i^*\partial^\mu \psi_i - \psi_i \partial^\mu \psi_i^*) \]

With the free-particle solutions \( \psi(x) = Ne^{-ipx} \) the current becomes:

\[ j_{\mu}^i = eN_i^*N_f(p_f+p_i)^\mu \times \exp[-i(p_f-p_i)] = e(p_f+p_i)^\mu(\psi^*_f \psi_i); \quad \text{or, if } i=f: \quad j^\mu = 2e|N|^2 p^\mu. \]

c. Show that the probability density \( j^0 \) is not positive definite. Describe, and illustrate the Feynman-Stückelberg reinterpretation of the negative-energy solutions \( \psi(x) \) of the Klein-Gordon equation. (9 pts)

The time-like component of \( j^\mu \), with wavefunctions \( \psi = Ne^{-ipx} = Ne^{-iEt+ipx} \) is:

\[ j^0 = ie(\psi^* d^\mu \psi - \psi d^\mu \psi^*) = 2e|N|^2 E, \]

which can be either positive or negative because the only restriction on \( E \) comes from the K-G equation, namely \( E^2 = p^2 + m^2 \), or \( E = \pm \sqrt{(p^2 + m^2)} \).

Reinterpretation: negative-energy particle solution going backwards in time \( t \) is equivalent to a positive-energy antiparticle solution going forward in time: \( \exp[-iEt] \) with \( E<0 \) == \( \exp[-i(\bar{E})(-t)] \]

d. First, consider the scattering diagram: assume the exchange is via the exchange of a weak boson \( W^\pm \), whose propagator may be approximated at moderate energies to \( -ig_{\mu \nu}/M^2 \), with \( M = 80 \text{ GeV} \). Sketch the diagram and label all fourvectors. (6 pts)

\[ q \equiv p_3-p_1 \] interaction volume, i.e. region in space-time where \( \nu \neq 0 \).
e. Calculate the matrix element for this diagram in terms of CMS three-momenta and scattering angle $\theta(e^-,\nu_e)$. (14 pts)

For the process $a+b\rightarrow a'+b'$, with $m_a=m_a'$ and $m_b=m_b'$ we have found (assuming $a\neq b$):

$$m_{a+b\rightarrow a'+b'} = ig_w(p_a+p_a')\left(\frac{g_{\mu\nu}}{q^2-M^2}\right)ig_w(p_b+p_b')\mu = \frac{g_w^2}{M^2}(p_a+p_a')\mu(p_b+p_b')\mu = \frac{g_w^2}{M^2}(s-m_a^2-m_a'^2+(-u+m_a^2+m_a'^2)) = \frac{g_w^2}{M^2}(s-u)$$

In CMS variables:

$$s-u\big|_{m_a=m_a'=0} = 2p_ap_b + 2p_ap_{b'} + \frac{E_a=E_a'=p}{E_b=E_b'=p} = 4p^2 + 2p^2 - 2p^2 \cos(\pi-\theta) = 2p^2 (3 + \cos \theta)_{CMS}$$

f. Second, consider the annihilation diagram: assume the diagram is via the annihilation into a neutral $Z$ boson, with propagator $-ig_{\mu\nu}/M^2$. (We’ll assume the masses and couplings of the $W$ and $Z$ bosons to be equal for ease of calculation.) Sketch the diagram and label all fourvectors. (6 pts)

g. Calculate the total matrix element for $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$ in terms of CMS three-momenta and scattering angle $\theta(e^-,\nu_e)$. (14 pts)

For the process $a+b\rightarrow Z\rightarrow a'+b'$, with $m_a=m_a'$ and $m_b=m_b'$ we find (assuming $b=a$), using the previous matrix element calculation, and replacing $p_a\rightarrow -p_b$ and $p_b\rightarrow -p_a$:

$$m_{a+b\rightarrow a'+b'} = ig_w(p_a-p_a')\left(\frac{g_{\mu\nu}}{q^2-M^2}\right)ig_w(-p_a+p_b')\mu = \frac{g_w^2}{M^2}((-p_a+p_b')\mu(-p_a+p_b')\mu) = \frac{g_w^2}{M^2}(p_ap_b+p_ap_b'+p_ap_b') = \frac{g_w^2}{M^2}((-u+m_a^2+m_a'^2)+(t-m_a^2-m_a'^2)) = \frac{g_w^2}{M^2}(t-u)$$

In CMS variables:

$$t-u\big|_{m_a=m_a'=0} = 2p_ap_b - 2p_ap_{b'} = \frac{E_a=E_a'=p}{E_b=E_b'=p} = 2p^2 (1-\cos(\pi-\theta)) - 2p^2 (1-\cos \theta) = 4p^2 \cos \theta|_{CMS}$$

The total matrix element is the sum of the above, and the result from (e.), see (h.) below.
h. Calculate the differential cross section \( d\sigma /d\Omega^* \) for the process.

Note that
\[
\frac{d\sigma_{a+b\rightarrow a'+b'}}{d\Omega} = \frac{1}{64\pi^2s} |\mathbf{m}_{a+b\rightarrow a'+b'}|^2
\]
and ignore the masses of the electron and neutrino.

\((10 \text{ pnts})\)

For the total process \( a+b\rightarrow a'+b' \), with \( m_a=m_a' \) and \( m_b=m_b' \) we find (assuming \( \bar{b}=a \)), using

the previous matrix element calculations:
\[
\frac{d\sigma_{a+b\rightarrow a'+b'}}{d\Omega} = \frac{1}{64\pi^2s} |\mathbf{m}_{a+b\rightarrow a'+b'}|^2 = \frac{g_w^4}{M^4} |2p^2(3+\cos\theta) + 4p^2\cos\theta|^2 = \frac{g_w^4}{M^4} \left(6p^2(1+\cos\theta)\right)^2 = \frac{9s^2g_w^4}{M^4}\cos^4\frac{\theta}{2}
\]

In case where all masses can be neglected, the differential cross section becomes:
\[
\frac{d\sigma_{a+b\rightarrow a'+b'}}{d\Omega}_{\text{CMS, ignoring all masses}} = \frac{g_w^4}{64\pi^2sM^4}^{9s^2} \frac{\cos^4\frac{\theta}{2}}{4M^4}\cos^4\frac{\theta}{2}
\]

i. Considering that weak interactions violate parity, how would that show up in a correct calculation of the differential cross section? What possible functional form of the differential cross section would exhibit parity violation? \((8 \text{ pnts})\)

Parity violation in the annihilation process (see (f.)) would result in the situation where the angular distribution of the neutrino would be different from the distribution of the anti-neutrino, or different distributions of left-handed and right-handed neutrinos; i.e. that the distribution would NOT be symmetric around \( \theta=90^\circ \). The distribution we calculated in (f.) IS symmetric around \( \theta=90^\circ \). A parity violating term here might be an extra term of opposite (odd) parity, e.g. \( A\cos\theta \), because \( P(A\cos\theta) = A\cos(\pi-\theta) = -A\cos\theta \neq A\cos\theta \)

j. Calculate the total cross section in barn for \( s = 10 \text{ GeV}^2 \). \((8 \text{ pnts})\)

Integration gives:
\[
\sigma_{a+b\rightarrow a'+b'}|_{\text{CMS, ignoring all masses}} = \frac{g_w^4}{64\pi^2s} \frac{9s^2}{M^4} 2\pi \int d\cos\theta \frac{1}{4}(1+\cos\theta)^2 = \frac{9\pi\alpha_e^2s}{8M^4} \int_0^1 x^2 dx
\]
\[
= \frac{3\pi\alpha_e^2s}{8M^4} = 1.06 \text{ pb}
\]