PHY557-F09 – Final Examination

2.5 Hours, 4 Problems (one on back)! Start each problem on a new sheet of paper. Use only one side of the paper (your papers will be scanned). Before the end of the exam, number and initial (print!) each sheet you hand in.

I. Starting with the Dirac Lagrangian \( L_D = \bar{\psi}(x)[i\gamma^\mu\partial_\mu - m]\psi(x) \), where \( \bar{\psi} = \psi^\dagger \gamma^0 \), derive the Dirac equations for \( \psi \) and \( \bar{\psi} \) as the Euler-Lagrange equations of \( L_D \). (10 points)

Solution:
The Euler-Lagrange equations are:

\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) - \frac{\partial L}{\partial \psi} = \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_x \bar{\psi})} \right) - \frac{\partial L}{\partial \bar{\psi}} = 0
\]

Thus, for \( \psi \) and \( \bar{\psi} = \psi^\dagger \gamma^0 \) we find:

\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) - \frac{\partial L}{\partial \psi} = i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0, \quad \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_x \bar{\psi})} \right) - \frac{\partial L}{\partial \bar{\psi}} = -\gamma^\mu (i\gamma^\nu \partial_\mu - m)\psi = 0.
\]

II. Use the Feynman rules for the following processes: (60 points)

a. Draw and label ALL the dominant diagrams.

b. Mark each diagram as a weak (W), electromagnetic (EM), or strong (S) interaction diagram.

c. List the appropriate vertex factors, propagators, and in/out going particles according to the Feynman rules. What do you propose to do with quark lines that are contained within the same hadron (remember pion decay)?

Note: Do not do any calculations. For strong interaction processes, you may relax on tasks a and c.

<table>
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<th>Process</th>
<th>1. ( e^+ e^- \rightarrow e^+ e^- )</th>
<th>2. ( \gamma + e^- \rightarrow \gamma + e^- )</th>
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<td>3. ( \pi^- \rightarrow \mu^- + \nu_\mu )</td>
<td>4. ( K^{*-} \rightarrow \pi^+ + K^0 )</td>
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<td>5. ( \Lambda \rightarrow p + e^- \bar{\nu}_e )</td>
<td>6. ( K^0 \rightarrow \pi^+ + \pi^0 )</td>
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Solutions:

1. \( e^+ e^- \rightarrow e^+ e^- \): Two electroweak \( s \)-channel annihilation diagrams via (virtual) \( Z \) and photon, and two electroweak \( t \)-channel exchange diagrams with photon and \( Z \). The fermion-photon vertex couplings are \(-ie\gamma^\mu\) and the photon propagator is \(-ig_{\mu\nu}/q^2\). The photon diagrams dominate at low energy, but the \( Z \) diagram, which has the SM weak coupling with \( c_V \) and \( c_A \) and the \( Z \) boson propagator, dominates at CM energies around the \( Z \)-mass. There are also two diagrams with Higgs, but these will be very weak!

2. \( \gamma + e^- \rightarrow \gamma + e^- \): electromagnetic Compton scattering, see notes: two \( s \)-channel diagrams differing in the time ordering of the photon vertex: the two photons \( e^\pm_{1,2} \) are indistinguishable. The vertex factors are as above, but the propagator is now the fermion propagator …

3. \( \pi^- \rightarrow \mu^- + \nu_\mu \): the dominant weak decay of the pion, the lightest meson, see the lecture notes. The pion-\( W \) coupling could be written as part of a Lorentz 4-current, and we argued that this could be reduced to \( p^\mu f_\pi \), with the pion form factor \( f_\pi \) which represents the effects of the two quarks bound in the pion: \( M_{\pi^\pm} = f_\pi (G_F / \sqrt{2}) p_\mu V_{\mu\nu}(p_1)\gamma^\nu (1 - \gamma^5) u_\nu(-p_2) \), where the \( W \) boson propagator is contained in Fermi’s constant.
4. \( K^+ \rightarrow \pi^+ + K^0 \): this the strong decay of the \( J=1 \) strange meson, the “recurrence” of the \( J=0 \) kaons. There is NO change of strangeness here, another indication of the presence of the strong force only. The diagram has an “internal” gluon line off either one of the quarks in the \( K^* = u \bar{s} \), which splits into a \( \bar{d}d \); the \( \bar{d} \)-quark and the \( u \)-quark (from the \( K^* \)) form a \( \pi^+ \), while the \( d \)-quark and the \( s \)-quark (from the \( K^* \)) make a \( K^0 \).

5. \( \Lambda \rightarrow p \ e^- \ \bar{\nu}_e \): this the weak decay of the lowest-mass strange baryon; strangeness changes by one unit, and so does the charge of the baryon or meson involved: \( \Delta S=\Delta Q \). It is very similar to the more common decay \( \Lambda \rightarrow p \ \pi^- \) : the weak decay of the \( s \)-quark into a \( W^+ \) plus \( u \)-quark (thus involving the CKM matrix element \( V_{us} \)), followed by the decay of the virtual \( W^+ \) into a \( \pi^- \), or a \( e^- \ \bar{\nu}_e \) pair.

6. \( K^0 \rightarrow \pi^+ + \pi^- \): again we have a change of strangeness between initial and final state; this is a weak decay! The \( K^0 \) is a \( J=0 \ \bar{s}u \) meson. The \( s \)-quark decays into a \( W^+ \) plus a \( u \) (involving \( V_{us} \)), and the virtual \( W^0 \) into a \( \bar{d}u \) pair (involving \( V_{ud} \)). The original \( u \)-quark (a “spectator” quark), and the three final state quarks form the \( \pi^+ \) and \( \pi^- \). There are two diagrams, differing in origin of the \( u \)-quark.

III. Short Answer Questions: (80 points)

1. Two photons are measured to have energies \( E_1 \) and \( E_2 \) in the ATLAS detector, and their directions are separated by angle \( \omega \). Calculate the mass of the parent of the two photons.

**Solution:**
The invariant mass of the two-photon parent is the length of the sum of the photon 4-vectors. Without loss of generality, we may choose our coordinate system such that the 4-vectors are \( p_1 = (E_1, 0,0,E_1) \) and \( p_2 = (E_2,0, E_2\sin \omega, E_2\cos \omega) \).

The parent mass is: \( m = \sqrt{(p_1 + p_2)^2} = \sqrt{(2p_1p_2)} = \sqrt{(2E_1E_2 - 2E_1E_2\cos \omega)} = 2 \sin(\omega/2)\sqrt{E_1E_2} \).

2. Calculate the laboratory beam energy needed for making the 80.4 GeV \( W \) boson directly with an anti-electron-neutrino beam hitting atomic electrons (assumed stationary).

**Solution:**
The Lorentz invariant \( s \), defined as \( s \equiv (p_1 + p_2)^2 = p_1^2 = p_2^2 = M_W^2 \), equals the total energy squared in the CMS. In the Laboratory system, \( s = (p_1 + p_2)^2 \) with \( p_1 = (E, p, 0, 0) \) and \( p_2 = (m_e, 0, 0, 0) \). Thus: \( M_W^2 = s = (E + m_e)^2 - p^2 = E^2 + m_e^2 + 2Em_e - p^2 = m_e^2 + m_e^2 + 2Em_e \). Thus the minimum neutrino beam energy \( E \) necessary to produce a \( W \) boson equals: \( E = M_W^2 - (m_e^2 + m_e^2)/(2m_e) = 6325 \) TeV. This is, of course, an impossibly high neutrino beam energy in the laboratory!

3. Describe how to create a beam of anti-electron-neutrinos.

**Solution:**
Neutrinos can only be produced in weak interactions. Thus we must produce weakly decaying particles. Three such particles we know: the charged pion, Kaon, and the muon. Charged (and neutral) pions and Kaons are produced prolifically in strong interactions such as accelerator protons colliding with a metal target. Charged pions decay dominantly into a muon plus muon neutrino, and only 1 per \( 10^6 \) into electron plus electron neutrino. However, muons, although they live about 100 times longer than pions, do decay, and only into an electron plus electron neutrino and a muon neutrino. In particular, \( \mu^+ \rightarrow e^- \ \nu_e \ \bar{\nu}_\mu \), followed by a “hadron filter” (i.e. a dirt berm) that only leaves muons to decay. Muon decay produces the best source of anti-electron neutrinos. This requires a long decay tunnel!

Alternatively or in addition, negative Kaons may be selected: 28% of the time these decay into hadrons only (which are filtered out), 64% into \( \mu^- \ \bar{\nu}_\mu \), and 5% of the time into \( \pi^0 \ e^- \ \bar{\nu}_e \). In all cases, it is impossible to avoid large numbers of muon-neutrinos in the beam!
A more common source of anti-electron-neutrinos are nuclear reactors; however, these produce mostly very low energy neutrinos without much directionality.

4. Show in detail the symmetry principle responsible for, and how it leads to, the conservation of electrical charge. Use the Klein-Gordon equation as example.

Solution:
As argued in the lecture, conservation of electric charge is directly related to the invariance of the Lagrangian under global (i.e. non-local) unitary transformations of the type $U = e^{-iαQ}$. Take as example the Klein-Gordon Lagragian for a complex scalar field:

$$L = \frac{1}{2} \left( \frac{∂^2}{∂t^2} - \frac{∂^2}{∂x^2} \right) φ - m^2 φ^* φ = \frac{1}{2} \left( \frac{∂^2}{∂t^2} - \frac{∂^2}{∂x^2} \right) φ - m^2 φ^* φ = 0 ,$$

where $j^μ = iQ \left[ \frac{∂L}{∂(∂^μ φ^*)} - \frac{∂L}{∂(∂^μ φ)} \right] = iq \left( φ^* (∂^μ φ) - (∂^μ φ^*) \phi \right)$ is the conserved current for the charge $q$ of the scalar field $φ$.

5. Show, by explicit substitution, that a solution of the Dirac equation, $ψ(x) = u(p)e^{-ipx}$, is also a solution of the Klein-Gordon equation.

Solution:
The Dirac equation is $(∂^μ φ^μ + m^2)ψ = 0$ and $(∂^μ φ^μ + m^2)ψ^* = 0$. The derivative on $ψ(x) = u(p)e^{-ipx}$ acts on the exponent only: $∂^μ u(p)e^{-ipx} = -ip_μ u e^{-ipx}$. Thus $(∂^μ φ^μ + m^2)ψ = (-p_μ p^μ + m^2)ψ = -E^2 + p^2 + m^2 = 0$.

6. Enumerate all the “free” parameters in the Standard Model, in case of zero-mass neutrinos.

Solution:
The 3 couplings of $U(1)$, $SU(2)$, and $SU(3)$; $g$, $g'$, and $α_S$. The 2 parameters $μ$ and $λ$ of the Higgs potential. The 6 quark masses and the 3 charged lepton masses (in case of massless neutrinos). Finally, the 4 parameters in the CKM mixing matrix. A total of 18 parameters.

7. Argue why a precision measurement of the masses of the $W$ and the top quark give an indirect determination of the Standard Model Higgs boson mass.

Solution:
The $W$ boson mass at tree level is determined by the Higgs potential and is related to the $Z$ mass as $M_W = \cos θ_W$ or $M_Z^2 \left(1 - M_W^2 / M_Z^2 \right) = απ \left( G_F \sqrt{2} \right)$. At the one-loop level, radiative corrections come in that modify this simple relationship. The dominant radiative diagrams that modify this relationship involve a top-bottom fermion loop, and a $W$ plus Higgs loop. These radiative corrections, and therefore the $W$ mass, depend on the top quark mass and Higgs mass. Turning this around, a precision measurement of both the top quark and the $W$ boson masses will give an indirect determination of the Higgs boson mass in the framework of the SM.

8. Why is the QED coupling increasing with $Q^2$, while the weak and strong couplings are instead decreasing with $Q^2$? A qualitative answer suffices.

Solution:
The electric charge that is measured, depends on the shielding from vacuum polarization, which in turn depends on the $Q^2$ energy (or the inverse probing distance) of the measurement. We showed how the higher order loop diagrams contributing to the same final state as a tree level coupling produce infinities and $Q^2$ dependent terms that can be absorbed into a “renormalized” coupling. This coupling is no longer constant but depends on the energy of the measurement $ψ(Q^2)$ and on the fermion and boson masses en-
tering into the loops. As the probing energy becomes larger, more massive fermion and boson loops become “excited” and contribute to the shielding. For an abelian theory like QED, the renormalization works such that only fermion loops exist and the coupling increases with decreasing distance to the charge (higher $Q^2$), because the shielding becomes less and less. For non-abelian theories, loops of fermions as well as bosons exist (self couplings!), which contribute with opposite sign. The net effect is that for non-abelian theories the couplings actually decrease with decreasing distance.
IV. The Standard Model Higgs Boson. In this problem, consider a Standard Model Higgs boson with mass $M_H = 120 \text{ GeV}$ produced at the Large Hadron Collider (LHC). The LHC is a proton-proton collider of 14 TeV center-of-mass energy. The SM Higgs boson with this mass is produced dominantly in gluon-gluon interactions. A 120 GeV Standard Model (SM) decays dominantly into a bottom-antibottom quark pair. (150 points)

1. Draw the dominant Feynman diagram of the Higgs decay and label the lines.

**Solution:** (10 points)

![Feynman diagram of Higgs decay](image)

2. Show, referring to the SM Lagrangian, that the Higgs to fermion-antifermion coupling, summed over all colors, equals $N_c \frac{m_f}{M_W} \frac{e}{2 \sin \theta_w}$, where $m_f$ is the fermion's mass, $M_W$ the mass of the $W$ boson, $e$ the electromagnetic coupling, $N_c$ the number of colors (3 for quarks, 1 for leptons), and $\theta_w$ the Weinberg angle.

**Solution:** (15 points)

From Lecture Notes 12: the $Hf\bar{f}$ coupling is $g_f/\sqrt{2}$. The fermion mass is generated by the Higgs boson and is $m_f = g_f v/\sqrt{2}$. The vacuum expectation value of the Higgs field $v=246$ GeV is directly related to the $W$ boson mass $M_W=gv/2$ and the weak coupling $g$. The weak coupling $g$ is related to the electromagnetic coupling $e$ via the Weinberg angle: $g=e/\sin \theta_w$. Putting all this together:

$$\frac{g_f}{\sqrt{2}} = \frac{m_f}{v} \frac{g m_f}{2 M_W} = \frac{e}{2 \sin \theta_w} \frac{m_f}{M_W}$$

If the fermion is a quark and thus colored, we must sum the three different color final states, i.e. multiply by $N_c=3$.

3. Argue, based on part b, that the Higgs to $b\bar{b}$ is the dominant decay mode for a 120 GeV SM Higgs.

**Solution:** (10 points)

The bottom quark at $m_b = 5 \text{ GeV}$ is the heaviest fermion that the 120 GeV Higgs can decay into. At this mass, the Higgs cannot decay into $t\bar{t}$, nor into $WW$ or $ZZ$. The next heaviest fermion is a pair of $c$-quarks at $m_c=1.5$ GeV and $N_c=3$, or tau-leptons $m_\tau=1.68$ GeV, $N_c=1$. Thus, the Higgs decays about 90% of the time into a pair of $b$-quarks.

4. Calculate the tree-level matrix element $|M_{H \rightarrow b\bar{b}}|^2$ for the decay of a 120 GeV Higgs boson into a pair of $b$-quarks, and show that $|M_{H \rightarrow b\bar{b}}|^2 = N_c^2 \frac{g^2 m_b^2}{2 M_W^2} M_H^2 \left(1 - \frac{4 m_b^2}{M_H^2}\right)$. 

**Solution:** (10 points)

$$|M_{H \rightarrow b\bar{b}}|^2 = N_c^2 \frac{g^2 m_b^2}{2 M_W^2} M_H^2 \left(1 - \frac{4 m_b^2}{M_H^2}\right)$$
**Solution:** (30 points)

The amplitude for the decay $H \rightarrow b \bar{b}$ is easily calculated from the coupling and the $b$-quark spinors:

$$-iM_{H \rightarrow b \bar{b}} = -i \frac{g}{2} \frac{m_b}{M_W} \bar{u}(p_1)u(p_2)$$

Note, that line 1 is an antiquark going out.

The matrix element squared, summed over the fermion spins, and summed over all $N_c$ colors, is then:

$$\left| \frac{m_{H \rightarrow b \bar{b}}}{M_W} \right|^2 = N_c \frac{g^2}{4} \frac{m_b^2}{M_W^2} \sum_{s_1,s_2} \left| \bar{u}(p_1)u(p_2) \right|^2 = N_c \frac{g^2}{4} \frac{m_b^2}{M_W^2} \sum_{s_1,s_2} \left| \bar{u}_1 u_2 \right|^2$$

$$= \sum_{s_1,s_2} \left( \bar{u}_2 \right)_\alpha \left( u_1 \right)_\alpha \left( \bar{u}_1 \right)_\beta \left( u_2 \right)_\beta = \sum_{s_1} \left( \bar{u}_1 \right)_\alpha \left( \bar{u}_1 \right)_\beta = \sum_{s_2} \left( u_2 \right)_\beta \left( u_2 \right)_\beta$$

$$= \left( \not{p} - m_1 \right)_{\alpha \beta} \left( \not{p} + m_b \right)_{\beta \alpha} = \text{Tr} \left[ \left( \not{p} - m_1 \right) \left( \not{p} + m_b \right) \right]$$

$$= \text{Tr} \left[ \not{p}_1 \not{p}_2 \right] - \text{Tr} \left[ m_1 m_2 \mathbf{1} \right] = 4 p_1 \cdot p_2 - 4m_1m_2 = 2M_H^2 - 8m_b^2$$

Combining this we find the squared amplitude for the decay:

$$\left| \frac{m_{H \rightarrow f \bar{f}}}{M_W} \right|^2 = N_c \frac{g^2}{4} \frac{m_b^2}{M_W^2} \sum_{s_1,s_2} \left| \bar{u}_1 u_2 \right|^2 = N_c \frac{g^2}{4} \frac{m_b^2}{M_W^2} \left( 2M_H^2 - 8m_b^2 \right) = N_c \frac{g^2}{2} \frac{m_b^2}{M_W^2} M_H^2 \left( 1 - \frac{4m_b^2}{M_H^2} \right)$$

5. Starting from: $d\Gamma(H \rightarrow f \bar{f}) = \frac{1}{F} (2\pi)^4 \delta^4(p_H - p_1 - p_2) \frac{dp_1}{(2\pi)^3 2E_1} \frac{dp_2}{(2\pi)^3 2E_2} \left| \frac{m_{H \rightarrow f \bar{f}}}{M_W} \right|^2$, with flux factor $F = 2M_H$ and $m_1 = m_2 = m_f$, derive the expression $\Gamma(H \rightarrow f \bar{f}) = \frac{1}{16\pi M_H} \left| \frac{m_{H \rightarrow f \bar{f}}}{M_W} \right|^2$ for the decay width for $H \rightarrow f \bar{f}$, in terms of the kinematic variables and the spin-averaged matrix element squared $\left| \frac{m_{H \rightarrow f \bar{f}}}{M_W} \right|^2$ for the process.

**Solution:** (30 points)

The SM Higgs is a spin-0 boson and has no structure. Its decay matrix element squared depends only on the Higgs mass, the fermion mass, and the $HF \bar{f}$ coupling, and is constant in this problem. Thus, the phase space integration can be done immediately. For illustration, we start with the decay width formula:
\[ \Gamma(H \rightarrow f_1 \overline{f}_2) = \frac{(2\pi)^4}{F} \left( \frac{1}{(2\pi)^6} \right) \int_{-\infty}^{\infty} \delta^3(p_1+p_2) \delta(E_1+E_2-M_H) \frac{dp_1 dp_2}{2E_1 2E_2} \left| m_{H \rightarrow f_1 \overline{f}_2} \right|^2 \]

\[ = \frac{1}{8\pi^2 M_H^2} \int d\mathbf{p} \delta(E_1+E_2-M_H) \frac{4\pi p^2 dp}{4E_1 E_2} \delta(E_1+E_2-M_H) \]

\[ = \frac{1}{8\pi M_H^2} \left| m \right|^2 \int \frac{p^2 dp}{E_1 E_2} \delta(p) = \frac{1}{8\pi M_H^2} \left| m \right|^2 \frac{1}{16\pi M_H^2} \sqrt{1-\frac{4m_b^2}{M_H^2}} \]

Thus, the width into a pair of \( b \)-quarks is:

\[ \Gamma(H \rightarrow b \overline{b}) = \frac{1}{16\pi M_H^2} \left( 1 - \frac{4m_b^2}{M_H^2} \right)^{3/2} \left| m \right|^2 = N_c^2 \frac{g^2}{32\pi} \frac{m_h^2}{M_W^2} M_H^2 \left( 1 - \frac{4m_b^2}{M_H^2} \right)^{3/2} \]

6. The production of the Higgs is dominated by a two-gluon diagram, where the interaction of two gluons, one from each proton, produces the Higgs boson. Sketch the Feynman diagram for this process. Check the Standard Model Lagrangian: do gluons couple directly to a Higgs boson?

Solution: (15 points)
Because the Higgs is not colored, gluons do not couple directly to the Higgs boson. Gluons only couple to colored quarks, and therefore there must be an intermediate virtual quark loop between the gluons and Higgs. Because the Higgs couples to the quarks in proportion to their mass, the indirect gluon-gluon-Higgs coupling is dominated by the heaviest quark in the loop, i.e. the top-quark loop.

7. The cross section for \( gg \rightarrow H \) has the following formula:

\[ \sigma(g_1 g_2 \rightarrow H) = \frac{1}{N_g^2 M_H^2} \Gamma(H \rightarrow g_1 g_2) \delta(s-M_H^2) \text{, with } N_g = 8 \text{ and } \Gamma(H \rightarrow g_1 g_2) = \frac{\alpha_s^2 g^2}{512\pi^3} M_H^3 \]

and \( F \approx 2 \). The variable \( s \) is the center-of-mass energy squared of the gluons, \( N_g = 8 \) the number of QCD gluons, \( g \) the \( SU(2)_L \) coupling constant, \( M_H \) the Higgs boson mass, \( M_W \) the mass of the W boson, and \( \alpha_s \) the strong coupling constant. Based on part 6 above, explain specifically but qualitatively (not quantitatively), the reasons for the presence of \( \alpha_s^2 \), \( g^2 \), \( M_W^2 \), and \( M_H \) in the cross section, including their powers. Explain the overall dimension of the expression for the cross section.

Solution: (20 points)
The amplitude for the process has two gluon couplings: \( \langle \alpha_s^2 \rangle = \alpha_s^2 \); i.e. \( \alpha_s^2 \) in the cross section. The intermediate loop is dominated by the (virtual) top quark, which couples to the Higgs boson as \( 2gm_t/M_H \), where the \( m_t \) is part of the factor \( F \), which explains the \( M_W^2 \). Finally, the Higgs mass \( M_H \) comes in as a combination of flux factor and the matrix element squared.

8. Argue why you would be able to discover a 120 GeV mass Standard Model Higgs boson better via its radiative decay into two photons, rather than via its direct decay into a bottom-antibottom quark pair. Try to give order-of-magnitude arguments.
Solution: (20 points)
The Higgs decay into a pair of (colored) quarks shows up in the detector as a pair of $b$-quark jets. These jets may be “tagged” by displaced $B$-decay vertex, or by the presence of a muon in the $B$-decay. However, there is an enormous physics background from direct, strong, production of $b$-quark pairs. This background is $10^{4-5}$ higher than the signal for low-mass Higgs. Thus, in order to see a signal, it will have to be extremely narrow, so that it will stand out over the background as a clear peak. This is impossible with jets, where the resolution is about $100\% / \sqrt{E_T}$.

It is possible with photons, which are colorless and are measured about 10 times more accurate than jets. The background to high energy photons are jets that “fake” a photon or have a single leading $\pi^0$, and real photons. A jet faking a photon or producing a single leading $\pi^0$ is suppressed by a factor of several thousand per jet at medium high energy. Thus, the radiative Higgs decay into photons, though much less favored, is much narrower and sitting on a physics background that is smaller.