Elementary Particle Physics
Fall 2009
Lecture 12

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A QED-like Model with a Complex Higgs Field $\phi$

Lagrangian: \[ L_{QED} = T_\phi + T_{QED} - V(\phi) = \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \left( -\frac{1}{2} \mu^2 |\phi|^2 + \frac{1}{4} \lambda^2 |\phi|^4 \right) \]

with $\phi$ a complex scalar field, and $D_\mu \equiv \partial_\mu + iqA_\mu$, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

$\phi(x)$ and $A^\mu(x)$ transform under local phase/gauge transformations as:

$\phi(x) \quad A^\mu(x) \quad \rightarrow \quad \phi'(x) = e^{i\alpha(x)} \phi(x) \quad A'^\mu(x) = A^\mu(x) - \partial^\mu \alpha(x)/q$

It is clear that the above Lagrangian is invariant under such transformations by construction; it contains a massless photon.
Now observe what happens when we consider small deviations around the \textit{true minimum}.

The locus of minima forms a full circle (dashed red) in the complex plane of the complex generalized coordinate $\phi(x)$, with a radius $v = |\mu/\lambda|$.

The minima are \textit{degenerate}:

Oscillations \textit{along the valley bottom} cost no energy and are thus \textit{massless}.

But, oscillations \textit{perpendicular to the valley floor} are experiencing a $+\text{ve}$ parabolic potential and will lead to a \textit{mass term}.

As before, we will pick a convenient vacuum expectation value, $\phi = v$ (real), and consider small oscillations $\phi'(x) = \phi(x) - v$ around this vacuum.

The complex field $\phi'(x)$ consists of two independent \textit{real} fields, $\eta(x)$ and $\zeta(x)$, so that $\phi'(x) \equiv \eta(x) + i\zeta(x)$.

For small $|\phi'(x)| \ll v$, we may write:

$$\phi(x) \equiv v + \phi'(x) = (v + \eta(x) + i\zeta(x)) \approx (v + \eta(x)) e^{i\zeta(x)/v} \quad \text{for} \quad |\eta(x)|, |\zeta(x)| \ll v$$

Note that indeed the (phase) angle of $\phi$ is $\approx \zeta/v$. 

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\( \phi(x) \equiv v + \phi'(x) = (v + \eta(x) + i\zeta(x)) \approx (v + \eta(x))e^{i\zeta(x)/v} \quad \text{for} \quad |\eta(x)|, |\zeta(x)| \ll v \)

Substituting this expression into the Lagrangian, we expect to see terms appearing in \( \eta(x) \) and \( \zeta(x) \).

However, \( \zeta(x) \) enters \textbf{only as a phase factor} \( \exp(i\zeta(x)/v) \).

Because the Lagrangian is \textit{explicitly invariant under local phase rotations}, we may actually use the local gauge freedom to \textit{rotate the phase factor away} by the gauge transformation:

\[
\begin{align*}
\phi & \quad \rightarrow \quad \phi'' = e^{-i\zeta/v} \phi \\
A^\mu & \quad \rightarrow \quad A'^\mu(x) = A^\mu(x) + \partial^\mu \zeta(x)/qv
\end{align*}
\]

Thus, field \( \zeta(x) \) will be \textbf{completely absent} from the Lagrangian!

Note that the choice of minimum (vacuum) \( \phi_0 = v = \text{real} \), is not so special; after all, any other vacuum expectation value can be reached simply by making an appropriate phase rotation in the weak isospin space.

Let us see the other consequences that result from the change in perspective – looking at \( \phi \) as deviations \( \phi'' \) from the minimum \( v \).

Without loss of generality we make the replacement \( \phi \rightarrow \phi'' = v + \eta(x) \) in \( L \):
Let us see the other consequences resulting from the change in perspective – looking at $\phi$ as deviations $\phi''$ from the minimum at $v$.

$$
\phi \atop A^\mu \to \left\{ \begin{array}{l}
\phi'' \equiv e^{-i\zeta/v} \phi \approx e^{-i\zeta/v} (v + \eta(x)) e^{i\zeta(x)/v} = (v + \eta(x)) \\
A''^{\mu}(x) = A^\mu(x) + \partial^\mu \zeta(x)/qv
\end{array} \right.
$$

Without loss of generality we may thus make the replacement $\phi \to v + \eta(x)$ in $L$:

$$
L = \frac{1}{2} (D^\mu '' \phi')^*(D^\alpha '' \phi') - \left( -\frac{1}{2} \mu^2 \Bigr|\phi''\Bigr|^2 + \frac{1}{4} \lambda^2 \Bigr|\phi''\Bigr|^4 \right) - \frac{1}{4} F^\nu_\mu '' F''^\mu\nu = 
$$

$$
= \frac{1}{2} \left( \partial^\mu + iq \left( A^\mu + \partial^\mu \zeta / qv \right) \right) (v + \eta(x)) \Bigr| (v + \eta(x)) \Bigr|^2 + \frac{1}{2} \mu^2 \left( v + \eta(x) \right)^2 - \frac{1}{4} \lambda^2 \left( v + \eta(x) \right)^4 - \frac{1}{4} F^\nu_\mu F''^\mu\nu = 
$$

$$
= \frac{1}{2} \left( \partial^\mu + iq A''^\mu \right) (v + \eta(x)) \Bigr| (v + \eta(x)) \Bigr|^2 - \mu^2 \Bigr( \eta^3 / v + \eta^4 / 4v^2 - \frac{v^2}{4} \Bigr) - \frac{1}{4} F^\nu_\mu F''^\mu\nu = 
$$

$$
= \frac{1}{2} \left( \partial^\mu \eta \right)^2 + \frac{1}{2} q^2 \left( A''^\mu \right) (v + \eta)^2 - \mu^2 \eta^2 - \mu^2 \Bigr( \eta^3 / v + \eta^4 / 4v^2 - \frac{v^2}{4} \Bigr) - \frac{1}{4} F^\nu_\mu F''^\mu\nu = 
$$

$$
= \frac{1}{2} \left( \partial^\mu \eta \right) (\partial^\mu \eta) - 2 \mu^2 \eta^2 \right) - \frac{1}{4} F^\nu_\mu F''^\mu\nu + \frac{1}{2} q^2 \nu^2 A''^\mu A''^\mu + \frac{1}{2} q^2 \eta^2 A''^\mu A''^\mu + q^2 \nu \eta A''^\mu A''^\mu - \frac{1}{4} F^\nu_\mu F''^\mu\nu
$$

K-G equation for massive $\eta$

Photon mass

Triple and Quartic $\eta$-Photon couplings

$\eta$ self couplings
Toy Model: QED with a Complex Higgs Field $\phi$

In conclusion:

- we find that if we break the symmetry (by considering the field $\phi$ with respect to a chosen minimum) \textit{in a theory that exhibits local gauge invariance}, the initially massless gauge field (the field $A_\mu(x)$) \textbf{acquires mass}.

- In return, one of the scalar fields \textit{disappears} (the “tangential” field $\xi(x)$), leaving a \textbf{single real, massive, scalar} (Higgs) \textit{radial} field $\eta(x)$:

  \[
  \eta\text{-field, (real), mass: } \sqrt{2\mu^2} \\
  \xi\text{-field disappers; is 'rotated away'} \\
  A_\mu\text{-field, acquires mass: } qv
  \]
**Symmetries of the Standard Model: $U(1)$, $SU(2)$, $SU(3)$**

We’ve now constructed a powerful toolset:

- we know how to describe **free** elementary particles: bosons with the Klein-Gordon Lagrangian; and fermions with the Dirac Lagrangian.

We introduce **interactions** by the use of the gauge principle: requiring invariance of the Lagrangian under local phase/gauge transformations of the particle fields, we are led to introduce ‘compensating gauge fields’ of the proper form.

The **gauge fields have to be massless** in order to preserve gauge invariance, which is clearly a problem when considering weak interactions where the gauge bosons are massive.

However, we can overcome this problem by invoking the **Higgs mechanism**: postulating the existence of a boson field with an odd shape, i.e. a non-zero expectation value, and by re-expressing the Higgs field with respect to a true minimum (vacuum), we break the manifest symmetry of the Lagrangian, but reap great benefits as well: **the gauge boson acquires mass**, just what we need for the weak gauge bosons.

A final benefit from the use of a theory with local gauge invariance is that such theories are inherently self-consistent when higher-order diagrams are considered. Very elegant cancellations between diagrams occur, which cure divergences that would otherwise make the theory meaningless.
SU(2)\textsubscript{L} of Weak Isospin: Weak Iso-doublets

We will require the weak Lagrangian to be invariant for gauge transformations of the SU(2) of weak isospin (rotations in the weak isospin space):

\[
\begin{pmatrix}
    v_e \\
    e^-
\end{pmatrix}_L \rightarrow \begin{pmatrix}
    v'_e \\
    e'^-
\end{pmatrix}_L = e^{i\sigma \beta(x)/2} \begin{pmatrix}
    v_e \\
    e^-
\end{pmatrix}_L = \exp \left\{ \frac{i}{2} \begin{pmatrix}
    i\beta_3 & i(\beta_1 - i\beta_2) \\
    i(\beta_1 + i\beta_2) & -i\beta_3
\end{pmatrix} \right\} \begin{pmatrix}
    P_L v_e \\
    P_L e^-
\end{pmatrix}
\]

with \(P_L \equiv \frac{1}{2}(1 - \gamma^5)\) acting on the Dirac spinors for the neutrino and for the electron, as before.

Note that for all weak isospin doublets, the upper member of the doublet must have one unit of charge more than the lower member!

For the SU(2) invariance to be established in the Lagrangian, we need to replace the regular derivative \(\partial^\mu\) everywhere by the SU(2) covariant derivative \(D^\mu \equiv \partial^\mu + \frac{1}{2}ig\sigma \cdot b^\mu\), where \(g\) is an arbitrary coupling constant.

The transformation of the three gauge fields \(b^\mu(x)\) is closely linked to the three fields \(\beta(x)\), because its transformation needs to cancel terms arising from derivatives of \(\beta(x)\):

\[
b^\mu' = Ub^\mu U^{-1} + \frac{i}{g} (\partial^\mu U) U^{-1}, \quad \text{with} \quad U \equiv e^{i\frac{1}{2}\sigma_\beta(x)}
\]

To implement the SU(2) invariance, we must replace \(\partial^\mu\) by \(D_\mu \phi = \left( \partial_\mu + \frac{ig}{2} \sigma \cdot b^\mu(x) \right) \phi\)
Mixing of the Strong Quark States in the Weak Interaction

A “slight” complication arises:
\[
\begin{align*}
\Lambda &\rightarrow p + W^-^*, \\
W^-^* &\rightarrow e^- + \nu_e \\
\end{align*}
\]
the existence of weak decays
\[
\begin{align*}
\Lambda &\rightarrow p + W^-^*, \\
W^-^* &\rightarrow \pi^- \\
\end{align*}
\]
of strange (charmed, bottom) particles, like:
\[
\begin{align*}
K^- &\rightarrow \mu^- + \nu_{\mu} \\
B^+(u \bar{b}) &\rightarrow D^0 (u \bar{c}) + W^+, \\
W^+ &\rightarrow e^+ + \nu_e
\end{align*}
\]
This implies “cross talk” between the generational doublets, implying that the quark doublets need modification:

Weak interaction eigenstates differ from the Flavor eigenstates of the strong and electromagnetic interactions!

The weak eigenstates are a mixture of the flavor eigenstates ...

Cabibbo proposed a unitary mixing matrix for the first two weak doublets:

\[
\begin{pmatrix}
u \\ d_C \\ s_C \\ \end{pmatrix}_L,
\begin{pmatrix}
c \\ \end{pmatrix}_L,
\begin{pmatrix}
d_C \\ s_C \\ \end{pmatrix}_L
\]
with
\[
\begin{pmatrix}
\cos \theta_C & \sin \theta_C \\
-\sin \theta_C & \cos \theta_C
\end{pmatrix}
\]
with \(\theta_C\) the Cabibbo mixing angle.

Experimentally: \(\sin \theta_C = 0.2257 \pm 0.0010\), i.e. the mixing between flavor generations is mild.

By convention, the \(u\)-type quarks are chosen unmixed: if not, an overall unobservable rotation applied to all quarks may unmix them. (Cabibbo, Phys. Rev. Lett. 10 (1963) 532.)
Mixing of the Strong Quark States in the Weak Interaction

Because the Cabibbo mixing matrix is unitary, it depends on only *a single real parameter* \((\theta_C)\).
Kobayashi and Maskawa expanded the matrix to *three generations*, including \(t\) and \(b\)-quarks.
The \(3\times3\) unitary Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix has *four real independent parameters*:

It may be constructed as a triple matrix product for rotational mixing between the

generations 1 and 2, 1 and 3, and 2 and 3, with *mixing angles* \(\theta_{12}\), \(\theta_{13}\), and \(\theta_{23}\), respectively,
and with *a single complex phase* \(\delta\):

\[
V_{\text{CKM}} = R(\theta_{23}) R(\theta_{13}, \delta) R(\theta_{12}) = \begin{pmatrix}
1 & \cdots & \cdots \\
\vdots & \ddots & \vdots \\
\vdots & \vdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
c_{13} & s_{13} e^{-i\delta} \\
\cdots & \cdots \\
-s_{23} & c_{23} \\
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} \\
\cdots & \cdots \\
-s_{12} & c_{12} \\
\end{pmatrix}
\]

Experimentally, diagonal CKM matrix elements are close to 1, and off-diagonal elements
small: \(s_{12} \approx \sin \theta_{12} = 0.2257 \pm 0.0010\), \(s_{23} \approx \lambda^2 A = 0.042 \pm 0.001\), \(s_{13} \approx \lambda^3 A (\rho + i\eta) = |0.0036 \pm 0.0002|\)

The CKM matrix is well approximated up to \(O(\lambda^4)\) by the Wolfenstein parametrization.
Why Four Parameters in the 3x3 Unitary CKM-Matrix?

The CKM matrix $V_{ff'}$ “rotates” the complex triplet of Mass eigenstates $d_f$, into the Flavor eigenstates $d_{f'}$ as:

$$d_f \Rightarrow V_{ff'} d_{f'}.$$

The matrix $V_{ff'}$ must conserve the norm of the complex quark wave functions and therefore is complex and unitary.

To describe any complex $N \times N$ matrix, one needs a total of $N^2$ complex or $2N^2$ real parameters. **Unitarity** provides $N^2$ constraints: $N$ real constraints of the type $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$, plus $N(N-1)/2$ complex constraints of the type $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ (Closure in the complex plane or “Unitarity Triangles”)

Thus, one needs a total of $2N^2 - (N+N(N-1)) = N^2$ parameters: $N^2 - N - N(N-1)/2 = N(N-1)/2 = 3$ real parameters plus $N^2 - N(N-1)/2 = N(N+1)/2 = 6$ complex phases; 9 parameters in total.

We may multiply all the quark states by a single global arbitrary and unobservable phase factor and reduce the independent phases in the CKM matrix by one.

Furthermore, we may use the four $(2(N-1))$ relative phases between the $N=3$ up-type quarks and between the $N=3$ down-type quarks to cancel four more complex phases.

In summary: we have $2N^2 - (N+N(N-1)) - 1 - 2(N-1) = N^2 - 2N + 1 = (N-1)^2$ independent parameters: 3 real parameters (e.g. mixing angles) and 1 complex irreducible phase.
Complex Phase and CP Violation

The complex phase allows for CP-violation in the quark decays.

CP-violation has been observed and measured in the neutral Kaon and $B^0$-meson systems. While very small in the Kaon system, it is rather large for the $B^0$-mesons. However, it turns out that this CP-violation is insufficient to explain, by itself, the observed dominance of matter over anti-matter in the universe.

If there would be more than three generations of quarks, the CKM matrix would have to be expanded correspondingly.

Therefore, the experimental measurement of the unitarity of the 3×3 matrix provides an important test for the SM with 3 generations.

For example, unitarity requires:

\[
VV^\dagger = 1 \quad \Rightarrow \quad V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0
\]

\[
\Rightarrow \quad \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} \approx \frac{A\lambda^3(\rho+i\eta)}{-A\lambda^3} + 1 + \frac{A\lambda^3(1-\rho-i\eta)}{-A\lambda^3} = 0
\]

This describes a closed “unitarity triangle” in the complex plane.
Experiments

The closure, i.e. the existence of only three generations, is tested by a wide variety of experiments and found to be well satisfied within current experimental precision.

The labels at each contour indicate the measured quantity; e.g. the label “α” indicates the experiment measured the angle α of the unitarity triangle.

\[
\frac{A\lambda^3 (\rho + i\eta)}{-A\lambda^3} + 1 + \frac{A\lambda^3 (1 - \rho - i\eta)}{-A\lambda^3} = 0
\]

95% CL measurement contours of the unitarity triangle exhibiting the complex CP-violating phase (Fig. 11.2, PDG, Rev. of Part. Phys., PL B667 (2008) 1.)
**SU(2)$_L$ Doublets and Singlets**

With some effort, massive neutrinos can be incorporated into the Standard Model in the same way as massive quarks, by coupling to the Higgs.

Similar to the mixing in the “quark-sector”, **lepton mixing** can be incorporated in the lepton-sector if neutrinos have mass. See the discussion in “Review of Particle Physics”

This also implies the existence of heavy right-handed neutrino singlets; heavy, in order to suppress their production. However, if present, such heavy right-handed neutrinos may have been produced in the very early universe, and may have survived to travel the universe without interacting (“relic neutrinos”).

**The right-handed leptons and quarks are all singlets under SU(2)$_L$: they all have weak isospin zero.**

Note that for all weak isospin doublets, the upper member of the doublet must have one unit of charge *more* than the lower member!
The Radiation term in the SU(2) Lagrangian

In close analogy to QED, we add to the Lagrangian a so-called radiation term of the form

\[ L_{\text{rad},SU(2)} = -\frac{1}{4} E_{\mu \nu} E^{\mu \nu}, \]

with \( E_{\mu \nu} \equiv D^\mu B^\nu - D^\nu B^\mu \), and \( D^\mu \) the covariant derivative and \( B^\mu \equiv \frac{1}{2} \sigma \cdot b^\mu \).

\( B^\mu \) is a 2×2 matrix operator in SU(2) space, and \( b^\mu \) is the set of three fourvector (gauge) fields, the weak SU(2) vector bosons.

Then:

\[
E_{\mu \nu} \equiv D_\mu B_\nu - D_\nu B_\mu = (\partial_\mu + igB_\mu)B_\nu - (\partial_\nu + igB_\nu)B_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu + ig(B_\mu B_\nu - B_\nu B_\mu)
\]

\[
= \partial_\mu B_\nu - \partial_\nu B_\mu + ig[B_\mu, B_\nu]
\]

\[
= \frac{1}{2} \partial_\mu (\sigma \cdot b_\nu) - \frac{1}{2} \partial_\nu (\sigma \cdot b_\mu) + ig \left[ \frac{1}{2} \sigma \cdot b_\mu, \frac{1}{2} \sigma \cdot b_\nu \right]_{[\sigma_i, \sigma_j] = 2i\epsilon_{ijk} \sigma_k} = \frac{1}{2} \sigma \cdot \left( \partial_\mu b_\nu - \partial_\nu b_\mu - g(b_\mu \times b_\nu) \right)
\]

i.e. clearly more complicated than the radiation term for QED!

The extra complication arises from the fact that we are working with (Pauli) matrices here, which do not commute:

the SU(2) symmetry is “non-Abelian”, whereas QED is Abelian.
The SU(3)_C of Color

It is now relatively straightforward to extend to higher-dimension gauge symmetries. The SU(3) of color symmetry is an example. The transformations under which we require the Lagrangian to be invariant are now taking place in three-valued color space.

SU(3) phase transformations U(x) are generated by the group of 3×3 Hermitian lambda matrices λ = λ_i, i=1..8, of which there are 8 independent ones: U(x) = exp{i½λ·g(x)}.

The transformation properties under local gauge transformations are fixed by the group’s structure constants: [λ_i,λ_j]=f^k_{ij}λ_k, where the f^k_{ij} are the structure constants.

The eight gauge boson fields G^i_μ(x), i=1..8, that need to be introduced in the covariant derivative are the eight gluons of Quantum Chromo-Dynamics (QCD).

Again, we cannot add mass terms for the gauge fields by hand; the gluons must be massless in order to preserve the local gauge invariance: **Gauge fields are massless in Yang-Mills theories.**

Note that in *non-Abelian Yang-Mills theories* the radiation term in the Lagrangian contains terms – e.g. the [B_μ,B_ν] before – that are powers of three and four in the gauge fields, hinting at the existence of 3-point and 4-point couplings between the gluons.

In Abelian QED such terms are necessarily absent.
The $U(1)_Y$ of Weak Hypercharge $Y_W$

Clearly charge is *not* a good quantum number for the weak doublets, because the members have different charge.

However, the quantity “Weak Hypercharge” $Y_W \equiv 2(Q - I^{(3)}_W)$ has unique values for the doublets:

$$Y_W \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = 2(Q - I^{(3)}_W)\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \begin{pmatrix} 2(0 - \frac{1}{2})\nu_e \\ 2(-1 - (-\frac{1}{2}))e^- \end{pmatrix}_L = (-1)\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$$

$$Y_W e^-_R = 2(-1)e^-_R = (-2)e^-_R, \quad Y_W \nu_{eR} = (0)\nu_{eR}$$

$$Y_W \begin{pmatrix} u \\ d \end{pmatrix}_L = 2(Q - I^{(3)}_W)\begin{pmatrix} u \\ d \end{pmatrix}_L = \begin{pmatrix} 2(\frac{2}{3} - \frac{1}{2})u \\ 2(-\frac{1}{3} - (-\frac{1}{2}))d \end{pmatrix}_L = (+\frac{1}{3})\begin{pmatrix} u \\ d \end{pmatrix}_L$$

$$Y_W u_R = 2(\frac{2}{3})u_R = (+\frac{4}{3})u_R, \quad Y_W d_R = (-\frac{2}{3})d_R$$

i.e. the right-handed neutrino does *not* carry weak hypercharge, and as such does not participate in the weak interaction!

We will impose the $U(1)_Y$ symmetry on the Lagrangian, similar to our earlier attempt to require symmetry under $U(1)$ of charge which gave rise to electromagnetism.
The Standard Model Lagrangian

The Standard Model Lagrangian is constructed to be locally invariant under the $U(1)_Y$ of weak hypercharge, the $SU(2)_L$ of weak isospin, and $SU(3)_C$ of color. Fermions are thus grouped in weak isospin doublets (and singlets) of spinors. Quarks are three-component color-spinor for QCD. Leptons are color-singlets.

For simplicity we will ignore the color group, because it does not affect our discussion on the electromagnetic and weak interactions: the color group is disjoint...

In order to make the Lagrangian invariant under local gauge transformations $U(x)$ of the particle fields, we are forced to introduce a “covariant derivative” $D_\mu$ of the proper form (the principle of minimal interaction):

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + \frac{ig'}{2} Y_w 1_2 a_\mu(x) + \frac{ig}{2} \sigma \cdot b_\mu(x) = \left( \partial_\mu + \frac{ig'}{2} Y_w a_\mu + \frac{ig}{2} b_\mu^{(3)} \right)$$

$$= \left( \partial_\mu + \frac{ig}{2} \left( b_{\mu}^{(1)} + ib_{\mu}^{(2)} \right) \right) + \left( \partial_\mu + \frac{ig'}{2} Y_w a_\mu - \frac{ig}{2} b_\mu^{(3)} \right)$$

This is the electroweak part of the covariant derivative in the Standard Model, with the $a_\mu(x)$ gauge vector field of the weak hypercharge symmetry, and the triplet of gauge vector fields $b_\mu(x)$ of the weak isospin symmetry.
The Electroweak Part of the SM Lagrangian

The full electroweak Lagrangian contains the following terms:

1. Free-particle (kinetic) terms for all weak fermionic doublets and singlets; interactions are specified by the replacement of derivatives by the covariant derivative.

   This way, the gauge fields are introduced with a specific form and couplings.

   The leptons and quarks obey the Dirac equation, and we expect the particle terms to mimic the Dirac Lagrangian:

   \[ L_{e,\nu_e} = \bar{\psi}_L i \gamma^\mu D_\mu \psi_L, \quad \text{with} \quad \psi_L = \psi_L(x) = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \begin{pmatrix} P_L \nu_e \\ P_L e^- \end{pmatrix}, \quad \nu_e, e^- \text{ Dirac spinors} \]

2. A free-particle, Klein-Gordon type, term for the Higgs field, together with the Higgs potential with the famous non-zero vacuum expectation value.

   The Higgs is a \( SU(2) \) doublet of complex scalar fields, i.e. four independent real scalar fields. The Higgs field, with a non-zero vacuum expectation value, will give mass to three of the \( SU(2) \) vector boson fields.

3. Free-field terms for the bosonic gauge fields.

   For the \( SU(2) \) part we take, in close analogy to the \( U(1) \) QED Lagrangian, the form \( L_{\text{rad},SU(2)} = -\frac{1}{4} E_{\mu\nu} E^{\mu\nu} \), with \( E^{\mu\nu} \equiv D_\mu B^\nu - D_\nu B^\mu \), where \( B^\mu \equiv \frac{1}{2} \sigma \cdot b^\mu \).

   \( B^\mu \) is a 2x2 matrix operator in \( SU(2) \) space. The \( D^\mu \) will give rise to self-interactions between the gauge bosons, as well as with fermions.
The Higgs Part of the SM Lagrangian

As for QED, we postulate a Lagrangian $L$ for a scalar Higgs field $\phi$ that is a $SU(2)$ weak isospinor:

$$L_{\text{Higgs}} = \left( \partial_\mu \phi \right)^\dagger \left( \partial^\mu \phi \right) - V,$$

with $\phi = \frac{1}{\sqrt{2}} \left( \phi_1 + i \phi_2 \right)$, and potential

$$V = \mu^2 |\phi|^2 + \frac{\lambda^2}{4} |\phi|^4 = \mu^2 \phi^\dagger \phi + \lambda^2 (\phi^\dagger \phi)^2.$$

For $\mu^2 > 0$ this describes an iso-doublet of complex fields, or four real fields $\phi_\mu$, with mass $\mu$.

The Lagrangian has **global gauge invariance** for $SU(2)$ transformations: $\phi \to \phi' = \exp \left( i \frac{1}{2} \sigma \cdot \beta \right) \phi$

**local gauge invariance** for $SU(2)$ transformations requires a set of three gauge fields $b_\mu(x)$ to form covariant derivative $D_\mu = \partial_\mu + ig \sigma \cdot b_\mu/2$, with transformation $b_\mu' = b_\mu - \partial_\mu \beta/g - (\beta \times b_\mu)$.

As before, we next consider a Higgs-type potential with negative parabola $\mu^2 < 0$.

We may choose the simplest vacuum, a real local minimum with $\phi_1 = \phi_2 = \phi_4 = 0$, $\phi_3 = v = |\mu|/\lambda$, and re-write the Lagrangian for departures $\phi' = v + \eta$ from this minimum.

This will generate a mass term for the field $\phi'$ (as well as for the three gauge bosons):

$$V(\phi') = - |\mu|^2 |\phi'^\dagger \phi'| + \lambda^2 \left( \phi'^\dagger \phi' \right)^2 = - \frac{1}{2} |\mu|^2 \left( v + \eta(x) \right)^2 + \frac{1}{4} \frac{|\mu|^2}{v^2} \left( v + \eta(x) \right)^4$$

$$= - \frac{1}{2} |\mu|^2 \left( v^2 + 2v\eta + \eta^2 \right) + \frac{1}{4} \frac{|\mu|^2}{v^2} \left( v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4 \right)$$

$$= - \frac{1}{4} |\mu|^2 |v^2 + |\mu|^2 |\eta|^2 + |\mu|^2 |v\eta|^3 + \frac{|\mu|^2}{4v^2} \eta^4$$

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We will see below that this will lead to the desired result: a theory with local gauge invariance which is “hidden” and with gauge vector fields that now have mass terms.

Note, that the lower member of the Higgs doublet, the physical vacuum, must be neutral because it must be invariant under the $U_Q(1)$ group of transformations of electromagnetism, $U=\exp[i\alpha(x)Q]$, to prevent the photon from coupling to it and acquiring a mass!
The $U(1) \times SU(2)$ Covariant Derivative

As we argued before, it is imperative for a realistic theory that the weak gauge bosons be massive. The Higgs mechanism will accomplish that.

The term that is of importance is the Klein-Gordon term for the scalar Higgs field doublet $\phi$ (which has weak hypercharge $+1$):

$$D_\mu \phi = \left( \partial_\mu + \frac{ig'}{2} a_\mu(x) Y_W 1_{2} + \frac{ig}{2} \sigma \cdot b_\mu(x) \right) \phi = \begin{pmatrix} \partial_\mu + \frac{ig'}{2} a_\mu Y_W + \frac{ig}{2} b^{(3)}_\mu & \frac{ig}{2} \left( b^{(1)}_\mu - ib^{(2)}_\mu \right) \\ \frac{ig}{2} \left( b^{(1)}_\mu + ib^{(2)}_\mu \right) & \partial_\mu + \frac{ig'}{2} a_\mu Y_W - \frac{ig}{2} b^{(3)}_\mu \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

Moreover, the Higgs iso-doublet members must have weak isospin $\pm \frac{1}{2}$, and the top member must be larger in charge than the bottom member by one unit of $e$.

Therefore, we can choose the Higgs doublet either as $\phi = (\phi^+, \phi^0)$ ($Y_W = +1$) or as $\phi = (\phi^0, \phi^-)$ ($Y_W = -1$).

Note, as argued before, the physical Higgs field must be neutral.
The $U(1)\times SU(2)$ Covariant Derivative

As was shown, the Higgs field may be re-written around the true vacuum, which may be chosen without loss of generality as $\phi^0 = (0,0)$, $\phi^0 = (0,v/\sqrt{2})$, with $v = |\mu|/\lambda = \text{real}$. We use the gauge invariance of the Lagrangian to “rotate away” three (!) scalar fields $\zeta_i(x)$, while the Higgs field turns into $[v + \eta(x)]/\sqrt{2}$.

To check, we can rotate “back” the with the rotation $\exp(+i\sigma \cdot \zeta/v)$:

$$
\phi = e^{i\sigma \cdot \zeta/v} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} \approx \left(1 + i\frac{\sigma \cdot \zeta}{2v}\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i\frac{\zeta_3}{2v} & i\frac{\zeta_1 - i\zeta_2}{2v} \\ i\frac{\zeta_1 + i\zeta_2}{2v} & 1 - i\frac{\zeta_3}{2v} \end{pmatrix} \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} \approx \frac{1}{\sqrt{2}} \left(\frac{(\zeta_2 + i\zeta_1)/2}{v + \eta - i\zeta_3/2} \right)
$$

to find again a set of four real fields, or two complex fields, we started out with...

The covariant derivative of the (neutral) Higgs field becomes:

$$
D_\mu \phi = D_\mu \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(D_\mu + \frac{1}{2} ig^* a_\mu Y_w + \frac{1}{2} igb^{(3)} \right) \begin{pmatrix} 0 \\ v^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(D_\mu + \frac{1}{2} ig^* a_\mu Y_w - \frac{1}{2} igb^{(3)} \right) \begin{pmatrix} 0 \\ v^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(D_\mu + \frac{1}{2} igb^{(-)} (v^0 + \eta^0) \right) \begin{pmatrix} 0 \\ v^0 \end{pmatrix} = \left(Y_w \eta^0 = +\eta^0 \right)
$$
Gauge Boson Masses

Taking $\mu = |\mu| > 0$, i.e. positive and real, and using the Higgs potential $V(\phi) = -\mu^2 |\phi|^2 + \lambda^2 |\phi|^4$, rewritten around the vacuum expectation value $v = \mu/\lambda$, the Lagrangian part for the Higgs field becomes:

$$L_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) = \frac{1}{2} (\partial_\mu \eta)(\partial^\mu \eta) + \frac{g^2}{8} b^{(+)}_\mu b^{(-)}_\mu (v + \eta)^2 + \frac{1}{8} (g' a_\mu - g b^{(3)}_\mu)^2 (v + \eta)^2 - V(v + \eta)$$

where we find mass terms for three of the four gauge bosons of $U_Y(1) \times SU_L(2)$.

These mass terms imply **three more degrees of freedom** for the gauge fields which – being massless – originally only had transverse polarizations, but now acquire a longitudinal polarization in addition! These degrees-of-freedom come from the three fields $\zeta_i$ which have disappeared from $L$ and were “absorbed” into the gauge fields.
The Z-boson and the Photon

One gauge field remains massless: \( L_{\text{Higgs}} = \cdots + \frac{1}{8} g^2 v^2 b_\mu^{(+)} b_\mu^{(-)} + \frac{1}{8} v^2 (g' a_\mu - gb_\mu^{(3)})^2 + \cdots \)

In the above we identify the massive \( W^\pm \) and \( Z^0 \) bosons as linear combinations of the \( a_\mu \) and \( b_\mu \) fields, while the Higgs does not couple to the state orthogonal to the \( Z \) (the photon) which therefore remains massless:

\[
\begin{align*}
W^\pm_\mu &\equiv \frac{1}{\sqrt{2}} b_\mu^{(\pm)}, \\
Z_\mu^0 &\equiv \frac{-g' a_\mu + gb_\mu^{(3)}}{\sqrt{g^2 + g'^2}} \\
A_\mu &\equiv \frac{ga_\mu + g'b_\mu^{(3)}}{\sqrt{g^2 + g'^2}} \\
b_\mu^{(3)} &\equiv \frac{g' A_\mu + gZ_\mu}{\sqrt{g^2 + g'^2}}
\end{align*}
\]

We identify (note the factor \( \frac{1}{2} \) for a correct neutral vector boson mass term like the photon):

\[
\begin{align*}
\frac{1}{8} g^2 v^2 b_\mu^{(+)} b_\mu^{(-)} &= \frac{1}{8} g^2 v^2 \left( \frac{1}{2} |b_\mu^{(+)}|^2 + \frac{1}{2} |b_\mu^{(-)}|^2 \right) = \frac{1}{8} g^2 v^2 \left( |W^-_\mu|^2 + |W^+_\mu|^2 \right) \Rightarrow M_{W^\pm} = \sqrt{\frac{g^2 v^2}{4}} = \frac{gv}{2} \\
\frac{1}{8} v^2 (g' a_\mu - gb_\mu^{(3)})^2 &= \frac{1}{8} (g^2 + g'^2)v^2 |Z_\mu^0|^2 \Rightarrow M_{Z^0} = \frac{1}{2} v\sqrt{g^2 + g'^2} = M_W \sqrt{1 + g'^2/g^2}
\end{align*}
\]

with the Weinberg angle \( \theta_W = \arctan(g'/g) \): \( M_W/M_Z = \cos \theta_W \), which defines the physical particles, the \( Z^0 \) and the photon, in terms of mixtures of the neutral fields \( b_\mu^{(3)} \) and \( a_\mu \).

In case of no mixing between the \( U(1)_Y \) and \( SU(2)_L \) neutral gauge bosons, we would have \( g' = 0 \). Then \( \theta_W = 0 \) and the \( W^\pm \) and \( Z^0 \) bosons would have had the same mass (and \( e = 0! \))
Gauge Boson Interactions

The interactions between the gauge fields are contained in the radiation term 

\[-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{4} E_{\mu\nu} E_{\mu\nu}\]

in the Standard Model Lagrangian.

Here \( E_{\mu\nu} \equiv D^\mu B^\nu - D^\nu B^\mu \), with the covariant derivative \( D^\mu \) and \( B^\mu \equiv \frac{1}{2} \sigma \cdot b^\mu \),

and \( F_{\mu\nu} \equiv D^\mu a^\nu - D^\nu a^\mu \), where \( a^\mu(x) \) is the \( U(1)_Y \) gauge field.

In terms of the physical gauge fields \( W^\mu, Z^\mu \) and the photon \( A^\mu \), we find:

\[
L_{WWZ} = -ig \cos \theta_W [ (W^\dagger_\mu W^\nu - W^\mu W^\dagger_\nu ) \partial_\mu Z_\nu + F_{\mu\nu} W^\mu Z^\dagger_\nu - F^\dagger_{\mu\nu} W Z^\mu W_\nu ]
\]

\[
L_{WWA} = -ie [ (W^\dagger_\mu W^\nu - W^\mu W^\dagger_\nu ) \partial_\mu A_\nu + F_{\mu\nu} W A^\mu W^\dagger_\nu - F^\dagger_{\mu\nu} W A^\mu W_\nu ]
\]

where \( F_{\mu\nu} W \equiv \partial_\mu W^\nu - \partial_\nu W^\mu \), \( W^\mu \) represents the \( W^- \) boson field, while \( W^\dagger_\mu \) represents the \( W^+ \).

In addition, four-point couplings appear: \( ZZWW, AAWW, ZAWW, \) and \( WWWW \), e.g.:

\[
L_{WWWW} = g^2/2 \ W^\dagger_\mu W^\nu [ W^\dagger_\mu W^\nu - W^\mu W^\dagger_\nu ].
\]

In the Standard Model, couplings between only the neutral gauge bosons, the photon and \( Z \), are absent!
Interactions between Gauge Bosons and Fermions

The interactions of the elementary fermions, leptons and quarks, arise naturally from Dirac terms in the Lagrangian: \( L_{\text{int}} = \bar{f} i \gamma^\mu D^\mu f \), with \( f \) a lefthanded weak isospin doublet of fermions or a righthanded singlet, and \( D^\mu \) the covariant derivative.

We will use the interaction terms to fix the coupling to the physical photon to the experimental value \( e \), where \( \alpha \equiv e^2/4\pi = 1/137 \).

Taking the electron-neutrino doublet as example \((Y_W \ell_L = -1 \ell_L, Y_W e_R = -2 e_R )\), we find:

\[
L_{\text{int}} = (\bar{v}_e e) \left[ i \gamma^\mu \left( \partial^\mu + \frac{i}{2} g \sigma \cdot b(x) \right) \left( \frac{v_e}{e} \right) + \bar{e}_R i \gamma^\mu \left( \partial^\mu + \frac{i}{2} g \sigma \cdot W^\mu \right) \left( \frac{v_e}{e} \right) \right] e_R
\]

\[
= (\bar{v}_e e) \left[ i \gamma^\mu \left( \partial^\mu + \frac{i}{2} g (g^2 + g')^2 Z^\mu \right) - \frac{i}{\sqrt{2}} g W^+ \left( \partial^\mu + \frac{i}{2} g \sigma \cdot W^\mu \right) \right] \left( \frac{v_e}{e} \right) e_R
\]

\[
= \text{kinetic terms for } e \text{ and } \bar{v}_e + \frac{\sqrt{g^2 + g'^2}}{2} v_L \gamma^\mu v_L Z^\mu + \frac{g}{\sqrt{2}} \left( \bar{v}_L \gamma^\mu e_L W^+ + \bar{v}_L \gamma^\mu v_L W^- \right)
\]

\[
+ \frac{g g'}{\sqrt{g^2 + g'^2}} \left( e_L \gamma^\mu e_L + e_R \gamma^\mu e_R \right) A^\mu + \frac{1}{\sqrt{g^2 + g'^2}} \left( \frac{g^2 - g'^2}{2} e_L \gamma^\mu e_L - g^2 e_R \gamma^\mu e_R \right) Z^\mu
\]
The EM and Weak Couplings in the SM

Fixing the **photon coupling**, we need \( e = \frac{g g'}{\sqrt{g^2 + g'^2}} = g \sin \theta_W \):

\[
L_{int} = \cdots + \frac{g g'}{\sqrt{g^2 + g'^2}} \left( \overline{e}_L \gamma^\mu e_L + \overline{e}_R \gamma^\mu e_R \right) A_\mu + \cdots = \overline{e} \gamma^\mu e
\]

The **W coupling** is fixed by the highly accurate measurement of the muon lifetime, which gives the Fermi constant \( G_F \) (measured with muons in a circular storage ring where they have a well known momentum):

\[
G_F = \frac{1}{\sqrt{2}} \left( \frac{g}{\sqrt{2}} \right)^2 \frac{1}{M_W^2} = \frac{g^2}{8 M_W^2} = \frac{1}{2 \nu^2}
\]

This implies, for instance, that \( \nu = 246 \text{ GeV} \).

It can be seen that the SM predicts the existence of weak “neutral currents” mediated by the \( Z^0 \), in addition to the weak “charged currents” mediated by the charged gauge bosons \( W^\pm \).

Neutral currents are not easily seen in decays (e.g. \( \mu^- \rightarrow \mu^- + Z^0^* \), \( Z^0^* \rightarrow \nu + \bar{\nu} \) violates energy-momentum conservation), and appear clearly only in interactions.

At low energy, only *neutrino* interactions are feasible for a study of weak neutral currents, because *electromagnetic* neutral-current events, mediated by the photon, drown out the weak interaction by many orders of magnitude!

Eventually, in 1973, a handful of neutral current events was seen at CERN.
Couplings to the Z Boson

The coupling between the electron and the Z boson can be written in terms of the vector and axial-vector couplings $c_V$ and $c_A$:

$$
\frac{1}{\sqrt{G^2 + g'^2}} \left( \frac{g^2 - g'^2}{2} e_L \gamma^\mu e_L - g'^2 e_R \gamma^\mu e_R \right) Z_\mu
$$

$$
= \frac{1}{\sqrt{G^2 + g'^2}} \left( \frac{g^2 - g'^2}{2} + g'^2 \right) e_L \gamma^\mu e_L - g'^2 e_R \gamma^\mu e_R \right) Z_\mu
$$

$$
= \frac{1}{\sqrt{G^2 + g'^2}} e \left( \frac{g^2 + g'^2}{2} P_L - g'^2 \right) \gamma^\mu e Z_\mu = \frac{1}{\sqrt{G^2 + g'^2}} \frac{g^2 + g'^2}{2} P_L - g'^2 \right) \gamma^\mu e Z_\mu
$$

$$
= -\frac{g}{2 \cos \theta_W} e \left( -\frac{1}{2} + 2 \sin^2 \theta_W + \frac{1}{2} \gamma^5 \right) \gamma^\mu e Z_\mu = \frac{g}{2 \cos \theta_W} \left( c_V - c_A \gamma^5 \right) \gamma^\mu e Z_\mu
$$

For the quarks, the coupling to the Z is similar:

the vector and axial-vector couplings of the Z to fermions in terms of the $U(1)_Y$ and $SU(2)_L$ groups are as the last expression in the above, with $c_V = I_3 - 2Q \sin^2 \theta_W$, $c_A = I_3$.  

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Summary SM Couplings

In summary, in the Standard Model, we find a number of “arbitrary” parameters: 
\( g, g', \mu^2, \) and \( v \).

Three of these: \( g, g', \) and \( v \); can be related to extremely well-measured experimental quantities: \( \alpha, M_Z=91,187.6 \pm 2.1 \text{ MeV}, G_F =1.16637 \times 10^{-5} \text{ GeV}^2 \).

Other parameters: \( \sin^2 \theta_W, M_W \) can then be predicted/calculated and compared to experiment as a check on the Standard Model.

The parameter \( \mu^2 \) is only related to the Higgs boson mass; until the Higgs is found, it remains essentially unknown.

However, consideration of higher order terms in Feynman diagrams tells us, that if the ‘t Hooft-Veltman cancellations are to work in higher order diagrams (renormalizability of the Standard Model), the Higgs mass cannot be very much higher than the weak gauge boson masses: \( m_H <1000 \text{ GeV} \).
Fermion Masses

Because the basic fields in the SM are the $SU(2)$ doublets and the singlets, simple Dirac mass terms of form $m\ell\ell$ are \textbf{not gauge invariant}: $\nu_e$ and $e$ have different masses!

Under $SU(2)$ weak isospin rotations, the Dirac spinor $e = (P_L e + P_R e) = (e_L + e_R)$ changes: the $e_L$ as part of a doublet mixes with the $\nu_e$, whereas $e_R$, a $SU(2)$ singlet, does not.

Again, the Higgs mechanism helps out: we introduce $SU(2)$-invariant interaction terms of the type (taking $m_\nu = 0$):

$$L_{\text{lepton-Higgs}} = -g_\ell \left[ \bar{\ell}_R \left( \phi^+ \ell_L \right) + \left( \bar{\ell}_L \phi \right) \ell_R \right] \text{, with } \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix}, \quad \ell_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad \ell_R = e_R$$

$$= -\frac{g_e}{\sqrt{2}} \left( v + \eta(x) \right) \left[ \bar{e}_R e_L + \bar{e}_L e_R \right] = -\frac{g_e}{\sqrt{2}} v \left[ \bar{e}_R e_L + \bar{e}_L e_R \right] - \frac{g_e}{\sqrt{2}} \eta(x) \left[ \bar{e}_R e_L + \bar{e}_L e_R \right]$$

$$m_e = \frac{g_e v}{\sqrt{2}}, \quad \frac{g_e}{\sqrt{2}} = \frac{m_e}{v} = m_e \sqrt{2G_F} \text{ Higgs to } e \text{ coupling}$$

Because $\phi^+ \ell_L$ and $\bar{\ell}_L \phi$ are $SU(2)$ scalars (dot-products of $SU(2)$ weak iso-spinors) such terms in $L_{\text{lepton-Higgs}}$ are scalars under $SU(2)_L$ transformations.

Also, the neutrino is left massless and does not couple to the Higgs!

The couplings of the charged leptons to the Higgs are “fixed” experimentally by the measured fermion mass: e.g. $g_e = m_e / v = 2.1 \times 10^{-6}$. 
Down-type Quark Masses

In this way, for each massive fermion we introduce an a-priori arbitrary coupling constant $g_\ell$, a total of 9 (or 12 with massive neutrinos) additional parameters! Rather unsatisfactory...

Quark masses are read off the Lagrangian as well.

For the down-type quarks:

$$L_{d-Higgs} = -g_d \left[ \overline{d_R} \left( \phi^+ q_L \right) + \left( \overline{q_L} \phi \right) d_R \right], \quad \text{with} \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix}, \quad q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$$

$$= -\frac{g_d}{\sqrt{2}} \left( v + \eta(x) \right) \left[ \overline{d_R} d_L + \overline{d_L} d_R \right] = -\frac{g_d}{\sqrt{2}} v \left[ \overline{d_R} d_L + \overline{d_L} d_R \right] - \frac{g_d}{\sqrt{2}} \eta(x) \left[ \overline{d_R} d_L + \overline{d_L} d_R \right]$$

$$m_d = \frac{g_d}{\sqrt{2}} v; \quad \frac{g_d}{\sqrt{2}} = m_d \sqrt{2} G_F$$

However, this leaves the up-type quarks massless (like the neutrino)!
Up-type Quark Masses

Again, SU(2) symmetry helps us out:

we construct a charge-conjugate Higgs doublet \( \phi_c \) in a way completely analogous to the construction of the SU(2) strong isospin anti-doublet:

\[
\phi_c = -i\sigma_2\phi^* = \begin{pmatrix} -\phi^0 \\ \phi^- \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} v + \eta(x) \\ 0 \end{pmatrix}
\]

Using \( \phi_c \), which has an upper component after the SU(2)\(_L\) symmetry breaking, we may give mass to the up-type quarks:

\[
L_{u-Higgs} = -g_u \left[ \bar{u}_R \left( \phi^\dagger_c q_L \right) + \left( \bar{q}_L \phi_c \right) u_R \right], \quad \text{with} \quad \phi_c = \begin{pmatrix} -\phi^0 \\ \phi^- \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} v + \eta(x) \\ 0 \end{pmatrix}, \quad q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L
\]

\[
= -\frac{g_u}{\sqrt{2}} \left( v + \eta(x) \right) \left[ \bar{u}_R u_L + \bar{u}_L u_R \right] = -\frac{g_u}{\sqrt{2}} v \left[ \bar{u}_R u_L + \bar{u}_L u_R \right] - \frac{g_u}{\sqrt{2}} \eta(x) \left[ \bar{u}_R u_L + \bar{u}_L u_R \right]
\]

\[
m_u = \frac{g_u}{\sqrt{2}} v, \quad \frac{g_u}{\sqrt{2}} = m_u \sqrt{2G_F}
\]

Clearly, this is a way to also account for neutrino masses if so needed...

Note, that the Higgs couples to a fermion in proportion to its mass, i.e. a 100 GeV Higgs decays mostly to a pair of \( b \)-quarks.
The “Full” Standard Model Lagrangian

We have now constructed the full SM Lagrangian:

\[ L_{SM} = L_{Higgs} + L_{rad} + L_{int} + L_{Higgs-fermion} \]

\[ L_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) \]

Interactions of Higgs with Gauge Fields: 3 Gauge Boson masses, and 1 Higgs mass

\[ -\frac{1}{4} E_{\mu\nu} E^{\mu\nu} \]

Interactions of Higgs with Gauge Bosons

\[ \bar{f} \left( i \gamma^\mu D_\mu \right) f \]

Interactions of fermions with Gauge Bosons

\[ -g_d \left[ \bar{d}_R (\phi^\dagger d_L) + (\bar{d}_L \phi) d_R \right] - g_u \left[ \bar{u}_R (\phi^\dagger q_L) + (\bar{q}_L \phi) u_R \right] \]

Interactions of Higgs with fermions: couplings and masses for up-type and down-type fermions

Note, that much more detail is still hidden in the Lagrangian and we have not yet included QCD explicitly!

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## Summary Feynman Rules

### Summary of what we’ve derived:

<table>
<thead>
<tr>
<th>External Lines</th>
<th>Vertex Couplings</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fermion:</strong></td>
<td><strong>Scalar→Scalar – Photon</strong></td>
</tr>
<tr>
<td></td>
<td>(-iQe(p_{in} - p_{out})^\mu)</td>
</tr>
<tr>
<td><strong>Gluon:</strong></td>
<td><strong>Fermion→Fermion - Higgs</strong></td>
</tr>
<tr>
<td></td>
<td>(-ig/2 m_f / M_W)</td>
</tr>
<tr>
<td><strong>Massive Scalar Boson:</strong></td>
<td><strong>Quark(a)→Quark(b)-</strong></td>
</tr>
<tr>
<td></td>
<td>(-ig_S/2 T_{ba}^\gamma \gamma^\mu)</td>
</tr>
<tr>
<td><strong>Massive Vector Boson:</strong></td>
<td><strong>Z^\mu-boson→Z^\nu-boson - Higgs</strong></td>
</tr>
<tr>
<td></td>
<td>((ig/\sqrt{2})\gamma^\mu V_{ff'}^{\nu} \cdot 1/2 (1-\gamma^5))</td>
</tr>
<tr>
<td><strong>Scalar:</strong></td>
<td><strong>Fermion→Fermion - Photon</strong></td>
</tr>
<tr>
<td></td>
<td>(-iQe\gamma^\mu)</td>
</tr>
<tr>
<td><strong>Photon:</strong></td>
<td><strong>W^\mu-boson→W^\nu - Higgs</strong></td>
</tr>
<tr>
<td></td>
<td>(ig_{\mu\nu}gM_W)</td>
</tr>
<tr>
<td><strong>Propagators</strong></td>
<td><strong>Fermion→Fermion - W^\mu</strong></td>
</tr>
<tr>
<td><strong>Fermion:</strong></td>
<td>(i(\gamma^\mu q_{\mu} + m) / (q^2 - m^2))</td>
</tr>
<tr>
<td><strong>Gluon:</strong></td>
<td>(i / [q^2 - m^2 + im \Gamma])</td>
</tr>
<tr>
<td><strong>Massive Scalar Boson:</strong></td>
<td>(i / [q^2 - m^2 + im \Gamma])</td>
</tr>
<tr>
<td><strong>Massive Vector Boson:</strong></td>
<td>((ig_{\mu\nu}gM_W / (1 - \sin^2 \theta_W))</td>
</tr>
<tr>
<td><strong>Scalar Boson:</strong></td>
<td>(1)</td>
</tr>
</tbody>
</table>
Testing the Standard Model

A great variety of tests and measurements*) now exists which validate the Standard Model of particle physics in exquisite detail and with high precision.

In particular, the electroweak sector is very well tested quantitatively because perturbative calculations can be made and precision rapidly increases with increasing order in the electroweak coupling constants.

This is not true so much for the QCD sector of the SM; the strong coupling constant actually becomes comparable to unity at energies/distances of the proton mass/size, and the theory becomes non-perturbative.

Over the past decades, Lattice gauge calculations have succeeded in calculating a number of non-perturbative quantities with good success, at the cost of very large computing power.

In the following sections, we discuss a number of measurements and tests that illustrate the extraordinary quantitative success of the SM.