Elementary Particle Physics
Fall 2009
Lecture 08

MW 12:50 PM – 2:10 PM, Physics D-122
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Example: \( a + b \rightarrow 1 + 2 \) in CMS

The momenta \(|p_i|\) and \(|p_f|\) of the initial and final state particles are fully determined by the total CMS energy and the particles’ masses:

\[
s = (E_1 + E_2)^2 = E_1^2 + E_2^2 + 2E_1E_2 = 2p^2 + m_1^2 + m_2^2 + 2\sqrt{(p^2 + m_1^2)(p^2 + m_2^2)} \quad \left( p \equiv |p_f| \right)
\]

thus:

\[
(p^2 + m_1^2)(p^2 + m_2^2) = \frac{1}{4} \left( s - 2p^2 - (m_1^2 + m_2^2) \right)^2
\]

\[
\rho^2 + p^2(m_1^2 + m_2^2) + m_1^2m_2^2 = \frac{1}{4} \left( s^2 + 4p^4 + (m_1^2 + m_2^2)^2 - 4sp^2 - 2s(m_1^2 + m_2^2) + 4p^2(m_1^2 + m_2^2) \right)
\]

solve for \( p^2 \):

\[
p^2 = \frac{1}{4s} \left( s^2 + (m_1^2 + m_2^2)^2 - 2s(m_1^2 + m_2^2) - 4m_1^2m_2^2 \right) = \frac{1}{4s} \left( s^2 + (m_1^2 - m_2^2)^2 - 2s(m_1^2 + m_2^2) \right)
\]

\[
= \frac{1}{4s} \left( s - (m_1 + m_2)^2 \right) \left( s - (m_1 - m_2)^2 \right) = \frac{1}{4s} \lambda(s,m_1^2,m_2^2) \quad \rightarrow |p_f|^2_{CMS} = \frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{2\sqrt{s}}
\]

and similarly, \( |p_i^*| = \sqrt{\lambda(s,m_a^2,m_b^2)/(2\sqrt{s})} \), with \( \lambda(s,m_a^2,m_b^2) \) the so-called “triangle” function.

Note that the triangle function is a Lorentz scalar!
Laboratory System $a + b \rightarrow 1 + 2$

Often the cross section is needed expressed in Laboratory system variables, e.g. for electron-proton scattering where the proton – the target $b$ – is at rest, i.e. $p_b = (m_b, 0)$.

In the CMS: $\lambda(s, m_a^2, m_b^2) = 4s|p_i^*|^2$.

Expressed in Laboratory quantities ($b$ at rest) it becomes:

$$\lambda(s, m_a^2, m_b^2) = \left( s - (m_a + m_b)^2 \right) \left( s - (m_a - m_b)^2 \right) = s^2 + (m_a^2 - m_b^2)^2 - 2s(m_a^2 + m_b^2)$$

$$= (m_a^2 + m_b^2 + 2p_a p_b)^2 + (m_a^2 - m_b^2)^2 - 2(m_a^2 + m_b^2 + 2p_a p_b)(m_a^2 + m_b^2) \quad \left( p_{a,b} \text{ 4-vect's} \right)$$

$$= 4(p_a p_b)^2 + 4(p_a p_b)(m_a^2 + m_b^2) + (m_a^2 + m_b^2)^2 + (m_a^2 - m_b^2)^2 - 2(m_a^2 + m_b^2 + 2p_a p_b)(m_a^2 + m_b^2)$$

$$= 4(p_a p_b)^2 + 2(m_a^4 + m_b^4) - 2(m_a^2 + m_b^2)^2 = 4 \left[ (p_{a,b}^\mu p_{a,b;\mu})^2 - m_a^2 m_b^2 \right]$$

↓ In laboratory:

$$= (2E_a m_b - 2m_a m_b) \left( 2E_a m_b + 2m_a m_b \right) = 4m_b^2(E_a^2 - m_a^2) = 4m_b^2p_a^2 |\text{LABSYS}|$$

In order to find the elastic scattering cross section in the lab, we first express the differential cross section in Lorentz invariant quantities, and the scattering angle in terms of $q^2 = t$:  

\[ q^2 = t = (p_b - p_a)^2 = (p_b - p_a)^\mu(p_b - p_a;\mu) \]
Laboratory System \( a + b \rightarrow 1 + 2 \)

In order to find the elastic scattering cross section in the lab, we first express the differential cross section in Lorentz invariant quantities, and the scattering angle in terms of \( q^2 = t \):

At the \( b,2 \) vertex:
\[
-2q \cdot p_b = -2(p_2 - p_b) \cdot p_b = 2m_b^2 - 2p_2 \cdot p_b = q^2
\]

At the \( a,1 \) vertex:
\[
q^2 \equiv (p_a - p_1)^2 = m_a^2 + m_1^2 - 2E_a E_1 + 2 |p_a| |p_1| \cos \theta
\]

\[
q^2 \equiv m_a = m_1 = m_b = m_2
\]

\[
q^2 \equiv m_a = m = m_b = m_2
\]

\[
q^2 \equiv m_a = m \approx 0
\]

\[
q^2 \equiv m_b = m_2
\]

\[
q^2 \equiv m_a = m_1
\]

solving for \( E_1 \): \[
q^2 \equiv -2(E_a - E_1)m_b = -4E_a E_1 \sin^2 \frac{\theta}{2} \]

\[
E_1 = \frac{E_a}{1 + \frac{2E_a \sin^2 \frac{\theta}{2}}{m_b}}
\]

\[
q^2 \equiv 2 |p^*|^2 (1 - \cos \theta^*) = -4 |p^*|^2 \sin^2 \frac{\theta^*}{2}
\]

i.e. for elastic scattering \( E_1 \) depends only on \( \theta \)!
\textbf{IN-elastic Scattering ...}

For inelastic scattering, where \( m_b \neq m_2 \), it is a function of \textbf{two independent variables},

- e.g. \( E_1 \) and \( \theta \),
- or \( E_1 - E_a \) and the dimensionless variable “Bjorken-\( x \)”: \( x \equiv -2(p_b \cdot q)/q^2 \);
  for elastic scattering \( x = 1 \), else \( 0 \leq x < 1 \).

It is now straightforward to express the LABSYS and CMS angles in terms of the appropriate variables:

\[
dt = 2|\p^*|^2 d\cos\theta^* = \frac{|\p^*|^2}{\pi} d\Omega^*, \quad dt = 2m_b dE_1 = 2m_b E_a \left( \frac{E_1}{E_a} \right)^2 \frac{E_a}{m_b} d\cos\theta = -\frac{E_1^2}{\pi} d\Omega
\]

from which we express the differential cross sections:

\[
\frac{d\sigma(a+b\rightarrow1+2)}{d\Omega^*} \bigg|_{\text{CMS}} = \frac{1}{64\pi^2 s} \left| \frac{p_f^*}{p_i^*} \right|^2 \left| m_{a+b\rightarrow1+2} \right|^2,
\]

\[
\frac{d\sigma(a+b\rightarrow1+2)}{d\Omega} \bigg|_{\text{LABSYS}} = \frac{1}{64\pi^2 m_b^2} \left( \frac{E_1}{E_a} \right) \left| m_{a+b\rightarrow1+2} \right|^2
\]
Example: the Decay $a \to 1 + 2$

Again, a simple example is illustrative.

Consider the decay of a particle $a$ into two final state particles 1 and 2 in the CMS of particle $a$, where $p_1 + p_2 = 0$ and $\sqrt{s} = E_a = m_a$:

$$\Gamma(a \to 1 + 2) = \frac{(2\pi)^4}{2m_a} \frac{1}{(2\pi)^6} \int \cdots \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta^3(p_1 + p_2) \delta(E_1 + E_2 - m_a) |m_{a\to 1+2}|^2 =$$

$$= \frac{1}{8\pi^2 m_a} \iint \frac{d^4 p_f}{4E_1 E_2} \delta(E_1 + E_2 - m_a) |m|^2 = \frac{1}{8\pi^2 m_a} \iint \frac{p^2 dp d\Omega}{4E_1 E_2} \left[ \frac{\partial(E_1 + E_2 - m_a)}{\partial p} \right]^{-1} \delta(p - p_f) |m|^2$$

$$= \frac{1}{32\pi^2 m_a} \iint \frac{p^2 dp d\Omega}{E_1 E_2} \left[ \frac{p}{E_1} + \frac{p}{E_2} \right]^{-1} \delta(p - p_f) |m|^2 = \frac{1}{32\pi^2 m_a} \frac{p_f}{E_1 + E_2} \iiint d\Omega |m|^2$$

$$\Gamma(a \to 1 + 2) = \frac{|p_f^*|}{32\pi^2 m_a} \iiint d\Omega^* |m_{a\to 1+2}|^2 = \frac{\sqrt{\lambda(m_a^2, m_1^2, m_2^2)}}{64\pi^2 m_a^3} \iiint d\Omega^* |m_{a\to 1+2}|^2$$

For a unity matrix element the decay width is simply $\Gamma(a \to 1 + 2) = |p_f^*|/(8\pi m_a^2)$, with dimension GeV$^{-1}$; i.e. the matrix element must have dimension GeV.
The Field Theories

Local Gauge (Phase) Invariance is a powerful guiding principle for building theories and finding the form of the interactions.

Local Gauge Invariance is necessary for renormalizability of the Theory.

All theories realized in nature are believed to be locally gauge invariant theories.
Gauge Theories

All true present-day theories are based on **local gauge (or phase) invariance**, i.e. the *phase* of the wavefunctions can be varied according to an *arbitrary space-time dependent function*.

The Lagrangian $L=L(\psi,\partial_\mu \psi)$ that describes the system keeps the same form under phase transformations of the type:

$$\psi(\mathbf{r},t) \rightarrow \psi'(\mathbf{r},t) = e^{i\alpha(\mathbf{r},t)}\psi(\mathbf{r},t)$$

i.e. if $\psi(\mathbf{r},t)$ is a solution, then $\psi'(\mathbf{r},t)$ is a solution too!

This phase transformation is called “**local**”; the phase depends on the local coordinates $(\mathbf{r},t)$. Transformations with constant phase (i.e. the same for all space-time points) are “**global**”.

The term “gauge” is inherited from attempts by Hermann Weyl to construct theories with invariance under local *scale* (“eich”) transformations as a basis for electromagnetism.

**Local Gauge Invariance** is a powerful guiding principle, and is **necessary for renormalizability** of the calculations. All theories realized in nature are locally gauge invariant theories.

One can construct more complicated versions of the phase rotation, e.g. $\exp\{i\sigma \cdot \mathbf{b}(x)\}$ which is a two-by-two matrix, as can be seen from the series expansion of the $\exp$ function:

$$e^{i\sigma \cdot \mathbf{b}} = 1 + i\sigma \cdot \mathbf{b} + \frac{(i\sigma \cdot \mathbf{b})^2}{2!} + \frac{(i\sigma \cdot \mathbf{b})^3}{3!} + \ldots$$

with $\sigma \cdot \mathbf{b} = \sigma_1 b_1 + \sigma_2 b_2 + \sigma_3 b_3 = \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

with the standard representation of the Pauli matrices.
Massless Vector Bosons

In the absence of sources and currents \( j^\mu = 0 \) the Coulomb potential \( V \) is zero (or constant), and equation

\[
j = j^j = \partial_\mu \partial^\mu A^j - \partial^j \partial_0 A^0 = \frac{\partial^2 A}{\partial t^2} - \nabla^2 A + \frac{\partial}{\partial t} \nabla V
\]

Becomes the wave equation for free photons: \( \nabla^2 A = \partial^2 A / \partial t^2 \).

With solutions: \( A(r,t) = a \varepsilon \cos(k \cdot r - \omega t) \), with amplitude \( a \) and polarization vector \( \varepsilon \).

For the free photon the Coulomb gauge condition \( \nabla \cdot A = 0 \) translates into \( k \cdot \varepsilon = 0 \) (or \( p \cdot \varepsilon = 0 \) with \( \hbar \)), i.e. a free photon has only two independent components transverse to its direction!

The fourvector field \( A^\mu \) represents the photon, the carrier of electromagnetic interactions.

\( A^\mu \) is a four-vector field: the photon is a spin-1 (vector) boson.

All force carriers are vector bosons (except for the spin-2 Graviton). In preparation for the discussion of massive vector bosons \( W^\pm \) (80 GeV) and \( Z^0 \) (91 GeV), we consider the addition of a mass term to the Lagrangian:

\[
L_{EM} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 A^\mu A^\mu
\]

A gauge transformation of \( A^\mu \rightarrow A^\mu'(x) = A^\mu(x) + \partial^\mu \chi(x) \) is clearly not leaving this Lagrangian invariant: new terms like \( A^\mu \partial_\mu \chi \) and \( \partial_\mu \chi \partial^\mu \chi \) are introduced by \( A^\mu \rightarrow A^\mu'(x) = A^\mu(x) + \partial^\mu \chi(x) \).

Thus, gauge invariance requires the free photon to be massless.

This is fine for electromagnetism, but is a problem for the weak vector bosons: if they also have to stay massless, then the theory is definitely not going to describe reality!
The Klein-Gordon Equation for a Complex Scalar Field

A complex scalar field can be likened to a simultaneous description of two real scalar fields describing particles of precisely the same mass $m$.

We will consider the symmetry that exists in the Lagrangian for such two equal-mass fields:

$$L = \frac{1}{2} (\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2) + \frac{1}{2} (\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2)$$

We could have started with a different set of fields $\phi_1'$ and $\phi_2'$, related to $\phi_1$ and $\phi_2$ by a rotation:

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix}$$

$$\downarrow$$

$$L = \frac{1}{2} (\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2)$$

1\textsuperscript{st} term: $\partial_\mu (\phi_1' \cos \alpha - \phi_2' \sin \alpha) \partial^\mu (\phi_1' \cos \alpha - \phi_2' \sin \alpha) + \partial_\mu (\phi_1' \sin \alpha + \phi_2' \cos \alpha) \partial^\mu (\phi_1' \sin \alpha + \phi_2' \cos \alpha) = \cos^2 \alpha \partial_\mu \phi_1' \partial^\mu \phi_1' + \sin^2 \alpha \partial_\mu \phi_1' \partial^\mu \phi_1' + \sin^2 \alpha \partial_\mu \phi_1' \partial^\mu \phi_1' + \cos^2 \alpha \partial_\mu \phi_1' \partial^\mu \phi_1' = \partial_\mu \phi_1' \partial^\mu \phi_1' + \partial_\mu \phi_2' \partial^\mu \phi_2'$

2\textsuperscript{nd} term: $\phi_1^2 + \phi_2^2 = (\phi_1' \cos \alpha - \phi_2' \sin \alpha)^2 + (\phi_1' \sin \alpha + \phi_2' \cos \alpha)^2 = \phi_1'^2 + \phi_2'^2$
The Klein-Gordon Equation for a Complex Scalar Field

We can also write the Lagrangian in terms of a complex scalar field \( \phi \) with the identification: 
\[
\phi = (\phi_1 + i \phi_2)/\sqrt{2}.
\]

The rotation \((\phi_1, \phi_2) \rightarrow (\phi'_1, \phi'_2)\) is equivalent to a global phase change:
\[
\phi' = \frac{1}{\sqrt{2}} (\phi_1' + i \phi_2') = \frac{1}{\sqrt{2}} (\phi_1 \cos \alpha + \phi_2 \sin \alpha - i \phi_1 \sin \alpha + i \phi_2 \cos \alpha) = \frac{1}{\sqrt{2}} e^{-i\alpha} (\phi_1 + i \phi_2) = e^{-i\alpha} \phi
\]
where we used the Euler relation \(e^{i\alpha} = \cos \alpha + i \sin \alpha\).

The Lagrangian in terms of \( \phi \) becomes:
\[
\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2) \text{ with } \phi_1, \phi_2 \text{ real} \Rightarrow \phi^* = \frac{1}{\sqrt{2}} (\phi_1 - i \phi_2) \text{ and: } \phi_1 = \frac{1}{\sqrt{2}} (\phi + \phi^*), \phi_2 = \frac{-i}{\sqrt{2}} (\phi - \phi^*)
\]
\[
\Rightarrow L = \frac{1}{2} \left( (\partial_\mu \phi \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - m^2 (\phi_1^2 + \phi_2^2) \right)
\]

1\text{st term}: \[
\left( \frac{1}{\sqrt{2}} \right)^2 \partial_\mu (\phi + \phi^*) \partial^\mu (\phi + \phi^*) + \left( \frac{-i}{\sqrt{2}} \right)^2 \partial_\mu (\phi - \phi^*) \partial^\mu (\phi - \phi^*) =
\]
\[
= \frac{1}{2} \left( \partial_\mu (\phi + \phi^*) \partial^\mu (\phi + \phi^*) - \partial_\mu (\phi - \phi^*) \partial^\mu (\phi - \phi^*) \right) = \partial_\mu \phi \partial^\mu \phi
\]

2\text{nd term}: \[
\phi_1^2 + \phi_2^2 = \left( \frac{1}{\sqrt{2}} \right)^2 (\phi + \phi^*)^2 + \left( \frac{-i}{\sqrt{2}} \right)^2 (\phi - \phi^*)^2 = \frac{1}{2} \left( (\phi + \phi^*)^2 - (\phi - \phi^*)^2 \right) = \phi \phi^*
\]
\[
\Rightarrow L = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* \right)
\]
The Klein-Gordon Equation for a Complex Scalar Field

Thus, the Lagrangian \( L = \frac{1}{2} \left( \partial_{\mu} \phi \partial^{\mu} \phi^* - m^2 \phi \phi^* \right) \) is invariant under \( \phi \rightarrow \phi' = e^{-i \alpha} \phi \).

Invariance of \( L \) under such a \textit{global} symmetry transformation requires the existence of a \textbf{conserved Noether current}. We will derive this from first principles.

For ease of derivation we’ll consider small phase rotations \( e^{-i \alpha} \), with \( |\alpha| << 1 \). Then:

\[
\phi' = e^{-i \alpha} \phi \approx (1 - i \alpha) \phi = \phi + \delta \phi, \quad \text{with} \quad \delta \phi \equiv -i \alpha \phi
\]

and:

\[
\phi^* = (e^{-i \alpha} \phi)^* = e^{+i \alpha} \phi^* \approx (1 + i \alpha) \phi^* = \phi^* + \delta \phi^*, \quad \text{with} \quad \delta \phi^* \equiv i \alpha \phi^*
\]

Invariance of \( L \) requires:

\[
\delta L = 0 = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial^{\mu} \phi)} \delta (\partial^{\mu} \phi) + (\phi \rightarrow \phi^*) = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial^{\mu} \phi)} (\partial^{\mu} (\partial^{\mu} \phi) + (\phi \rightarrow \phi^*) =
\]

\[
= \frac{\partial L}{\partial \phi} \delta \phi + \partial^{\mu} \left( \frac{\partial L}{\partial (\partial^{\mu} \phi)} \delta \phi \right) - \partial^{\mu} \left( \frac{\partial L}{\partial (\partial^{\mu} \phi)} \partial^{\mu} \delta \phi \right) + (\phi \rightarrow \phi^*) =
\]

\[
= \left[ \frac{\partial L}{\partial \phi} - \partial^{\mu} \left( \frac{\partial L}{\partial (\partial^{\mu} \phi)} \right) \right] \delta \phi + \partial^{\mu} \left( \frac{\partial L}{\partial (\partial^{\mu} \phi)} \delta \phi \right) + (\phi \rightarrow \phi^*) =
\]

\[
= 0 - i \alpha \partial^{\mu} \left( \frac{\partial L}{\partial (\partial^{\mu} \phi)} \phi \right) + (\phi \rightarrow \phi^*) = i \alpha \partial^{\mu} \left[ \frac{\partial L}{\partial (\partial^{\mu} \phi^*)} \phi^* - \frac{\partial L}{\partial (\partial^{\mu} \phi)} \phi \right] = 0
\]
The Klein-Gordon Equation for a Complex Scalar Field

The Lagrangian \( L = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* \right) \) is invariant under \( \phi \rightarrow \phi' = e^{-i\alpha} \phi \).

Invariance of \( L \) under such a \textit{global symmetry transformation} requires the existence of a conserved Noether current:

\[
\delta L = i\alpha \partial^\mu \left[ \frac{\partial L}{\partial (\partial^\mu \phi^*)} \phi^* - \frac{\partial L}{\partial (\partial^\mu \phi)} \phi \right] = 0
\]

The final quantity in square brackets is a \textit{covariant fourvector}, and is conserved:

\[
\partial^\mu j_\mu = 0, \quad \text{with} \quad j_\mu \equiv iq \left[ \frac{\partial L}{\partial (\partial^\mu \phi^*)} \phi^* - \frac{\partial L}{\partial (\partial^\mu \phi)} \phi \right]
\]

where we added some constant “charge” \( q \) in the definition of the \textit{conserved current}. 
Current Conservation

The **conserved four-current** for the complex scalar field Lagrangian becomes:

\[ j_\mu = iq[\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi]. \]

This current has interesting properties:

if the field \( \phi \) represents a particle with mass \( m \), then the field \( \phi^* \) represents another particle with exactly the same mass: it is natural to consider \( \phi^* \) to represent the **antiparticle**.

Indeed, under the interchange \((\phi \leftrightarrow \phi^*)\), the four-current **changes sign**: \( j_\mu \leftrightarrow -j_\mu \) indicating that the charge of the antiparticle must be **opposite** that of the particle.

This true for a generalized charge: i.e. for **every additive** quantum number: strangeness, charm, etc. as well as for electric charge.

An easy way to find the conserved current is to start with the Klein-Gordon equation(s):

\[ (\partial_\mu \partial^\mu + m^2) \phi = 0, \quad (\partial_\mu \partial^\mu + m^2) \phi^* = 0 \]

and left-multiply the first by \( \phi^* \) and the second by \( \phi \) and subtract:

\[ \phi^* (\partial^2 + m^2) \phi = 0 \]

\[ \phi (\partial^2 + m^2) \phi^* = 0 \]

\[ \phi^* \partial^2 \phi - \phi \partial^2 \phi^* = \partial_\mu \left[ \phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi \right] = 0, \]

i.e. **conservation of the four-current** \( j_\mu = iq[\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi]. \)
Current Conservation in the Schrödinger Equation

This is very much similar to what we did in QM with the Schrödinger equation to obtain the conservation of probability or probability current:

\[
\phi^* \left( +i \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 \right) \phi = 0
\]

\[
\phi \left( -i \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 \right) \phi^* = 0
\]

\[
\frac{i}{\hbar} \left( \phi^* \frac{\partial}{\partial t} \phi - \phi \frac{\partial}{\partial t} \phi^* \right) + \frac{1}{2m} \left( \phi^* \nabla^2 \phi - \phi \nabla^2 \phi^* \right) = 0
\]

\[
\frac{\partial}{\partial t} \left( \phi^* \phi \right) + \frac{1}{2m} \nabla \cdot \left( \phi^* \nabla \phi - \phi \nabla \phi^* \right) = 0
\]

\[
\Rightarrow \quad \frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j} = 0 \quad \text{with} \quad \rho \equiv \phi^* \phi, \quad \mathbf{j} \equiv \frac{-i}{2m} \left( \phi^* \left( \nabla \phi \right) - \left( \nabla \phi^* \right) \phi \right)
\]
Conserved Currents

K-G current: \( j_\mu = iq\left[ \phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi \right] \)

Schrödinger current: \( j \equiv -i \frac{1}{2m} \left( \phi^* (\nabla \phi) - (\nabla \phi^*) \phi \right) \), \( \rho \equiv \phi^* \phi \)

However, there is a fundamental difference between the “Schrödinger” current and the Klein-Gordon current: \( \rho = j^0 \) has to be positive definite in order to be identified with a probability density; negative probabilities do not make sense.

In the Schrödinger current this condition is satisfied by the identification \( \rho \equiv \phi^* \phi = |\phi|^2 = j^0 \).

In the K-G equation we find: \( j^0 = i \left[ \phi^* (\partial^0 \phi) - (\partial^0 \phi^*) \phi \right] \)

\[ \downarrow \text{ using solutions: } \phi = ae^{-ipx} = ae^{i(p\cdot x - Et)} \]

\[ j^0 = i |a|^2 (-2iE) = 2|a|^2 E \quad \text{(hence: } a = 1/\sqrt{2|E|} \text{)} \]

where the parameter \( E \) is not positive-definite because \( E = \pm \sqrt{p^2 + m^2} \).

Formally, solutions of the type \( \phi = ae^{-ipx} \) with negative \( E \) cannot simply be ignored because the set of base vectors spanning the Hilbert space would be incomplete.

Because of the negative energy solutions, the Klein-Gordon equation (which was considered by Schrödinger and later by Klein and Gordon) was first abandoned in favor of the Schrödinger equation.
Particle and Anti-Particles

Later still, after the Dirac equation became firmly established and when Dirac successfully tamed its negative energy solutions, the Klein-Gordon equation started a second life.

Ultimately, Ernst Stueckelberg and later Richard Feynman re-interpreted the negative-energy particles going forward in time as positive-energy antiparticles going backwards in time (i.e. reversing direction):

\[ e^{-ipx} = e^{-i(Et - p \cdot r)} \iff e^{-i((-E)(-t) - (-p) \cdot (-r))} = e^{-i(-p)(-x)} \]

We can consider what happens to the charge current: \( j_\mu = iq[\phi^*(\partial_\mu \phi) - (\partial_\mu \phi^*)\phi] \):

\( j^0 \) represents a charge density (instead of a probability density) and may be positive or negative.

For electrons of positive energy \( E_1 \) or negative energy \(-E_1\), and charge \(-e\) we have:

- electron with \( E = E_1 > 0 \), \( p = p_1 \): \( j^\mu = -2e|a|^2 p^\mu \), and:
  \[
  \begin{cases}
  j^0 = -2e|a|^2 E_1 < 0 \\
  \mathbf{j} = -2e|a|^2 \mathbf{p}_1
  \end{cases}
  \]

- electron with \( E = -E_1 < 0 \), \( p = p_1 \): \( j^\mu = -2e|a|^2 p^\mu \),

  and:
  \[
  \begin{cases}
  j^0 = -2e|a|^2(-E_1) = +2e|a|^2 E_1 > 0 \\
  \mathbf{j} = -2e|a|^2 \mathbf{p}_1 = +2e|a|^2(-\mathbf{p}_1)
  \end{cases}
  \]

  = \underline{\text{positron with positive energy and traveling backwards}}!
Local Gauge Invariance

Above, we saw the invariance of the complex scalar field Lagrangian under phase transformations of the type $\phi \rightarrow \phi' = e^{i\alpha} \phi$, with $\alpha$ a constant.

This is a so-called “global” gauge (phase) transformation. This means that we require that the fields all phase-rotate together everywhere and for all time.

This doesn’t seem very elegant:

imagine that there is no electromagnetic interaction and only the strong interaction existed; in that situation the proton $\phi_1$ and neutron $\phi_2$ would be indistinguishable and have the same mass, and any mixture of the two would be simply the same nucleon.

Someone in the US and someone in China might define different mixtures as their “nucleon” but we would expect that the same (strong interaction) theory would work well in either case.

Let us investigate what would happen to the Lagrangian and the equations of motion if we were to require invariance under local phase transformations:

$\phi(x) \rightarrow \phi'(x) = e^{i\alpha(x)} \phi(x)$, where $x$, as usual, is the space-time coordinate $x^\mu$. 
Local Gauge Invariance

Local gauge invariance: i.e. invariance of the Lagrangian under *local* phase transformations:

\[ \phi(x) \rightarrow \phi'(x) = e^{i\alpha(x)} \phi(x) \], where \( x \) is the space-time coordinate \( x^\mu \).

Of course we expect trouble in the Lagrangian: after a local phase transformation of the fields the derivative terms will act on *both* \( \alpha(x) \) and \( \phi(x) \), thereby leading to extra terms involving the scalar function \( \alpha(x) \).

Otherwise stated: the local phase factor does not simply “pass through” the differentiation operator:

\[
\partial^\mu \left(e^{i\alpha(x)} \phi(x)\right) = i \left(\partial^\mu \alpha(x)\right)e^{i\alpha(x)} \phi(x) + e^{i\alpha(x)} \left(\partial^\mu \phi(x)\right) = e^{i\alpha(x)} \left(i \left(\partial^\mu \alpha(x)\right) + \partial^\mu \right) \phi(x) \neq e^{i\alpha(x)} \left(\partial^\mu \phi(x)\right)
\]

i.e. \( \partial^\mu \phi \) transforms *differently* from \( \phi \) itself, which is the cause of the non-invariance: if they both would be transforming in the same way, we could simply divide out the local phase factor in the Euler Lagrange equation.
Local Gauge Invariance

As an example, consider the difference between a global and a local phase transformation on the conserved current, $j_\mu = iq[\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi]$:

Global phase $\alpha$  

\[ j\mu' = iq \left[ \phi^* (\partial_\mu \phi') - (\partial_\mu \phi'^*) \phi \right] = iq \left[ e^{-i\alpha} \phi^* \left( \partial_\mu (e^{i\alpha} \phi) \right) - \left( \partial_\mu (e^{-i\alpha} \phi^*) \right) e^{i\alpha} \phi \right] = iq \left[ e^{-i\alpha} \phi^* e^{i\alpha} (\partial_\mu \phi) - e^{-i\alpha} (\partial_\mu \phi^*) e^{i\alpha} \phi \right] = iq \left[ \phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi \right] = j_\mu \]

Local phase $\alpha(x)$  

\[ j\mu' = iq \left[ \phi^* (\partial_\mu \phi') - (\partial_\mu \phi'^*) \phi \right] = iq \left[ e^{-i\alpha} \phi^* \left( \partial_\mu (e^{i\alpha} \phi) \right) - \left( \partial_\mu (e^{-i\alpha} \phi^*) \right) e^{i\alpha} \phi \right] = iq \left[ e^{-i\alpha} \phi^* \left( i(\partial_\mu \alpha) e^{i\alpha} \phi + e^{i\alpha} (\partial_\mu \phi) \right) - \left( -i(\partial_\mu \alpha) e^{-i\alpha} \phi^* + e^{-i\alpha} (\partial_\mu \phi^*) \right) e^{i\alpha} \phi \right] = iq \left[ e^{-i\alpha} \phi^* e^{i\alpha} (\partial_\mu \phi) - e^{-i\alpha} (\partial_\mu \phi^*) e^{i\alpha} \phi \right] - 2q (\partial_\mu \alpha) \phi^* \phi = j_\mu - 2q(\partial_\mu \alpha) \phi^* \phi \neq j_\mu \]

In order to implement local gauge invariance of the Lagrangian, we will have to add cancellation terms in the Lagrangian that, under transformations, acquire additive terms that can cancel the underlined term.

This reminds us of the gauge freedom of the photon field $A^\mu$: the physics was unchanged if we shifted $A^\mu$ by the derivative of a scalar function of space-time, very much like the $\partial_\mu \alpha$ term that we need to cancel here...
Gauge Freedom of $A^\mu$ and Local Gauge Invariance

Let us try to see how the combination $(\partial^\mu + A^\mu)\phi$ transforms:

$$\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x), \quad \text{and:} \quad A^\mu(x) \to A^\mu'(x) = A^\mu(x) - \partial^\mu \chi$$

$$(\partial^\mu + A^\mu)\phi \to (\partial^\mu + A^\mu)'\phi' = e^{i\alpha} \left( i(\partial^\mu \alpha) + \partial^\mu \right) \phi + (A^\mu - (\partial^\mu \chi) ) e^{i\alpha} \phi = e^{i\alpha} \left[ \partial^\mu + A^\mu + i(\partial^\mu \alpha) - (\partial^\mu \chi) \right] \phi$$

i.e. almost success!

The replacement that works is the identification $\chi = i\alpha$, or the combination (adding $q$!):

$$\partial^\mu \to \partial^\mu + iqA^\mu \quad \text{(with} \quad A'^\mu = A^\mu - \partial^\mu \alpha/q) \quad \text{or:} \quad i\partial^\mu \to i\partial^\mu - qA^\mu \quad \text{or:} \quad p^\mu \to p^\mu - qA^\mu$$

which is the principle of minimal replacement (W. Pauli) giving the EM interaction:

Simultaneous transformations:

$$\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x), \quad A^\mu(x) \to A^\mu'(x) = A^\mu(x) - \partial^\mu \alpha/q$$

$$\left( \partial^\mu + iqA^\mu \right)' \phi' = e^{i\alpha} \left( i(\partial^\mu \alpha) + \partial^\mu \right) \phi + iq \left( A^\mu - (\partial^\mu \alpha) / q \right) e^{i\alpha} \phi = e^{i\alpha} \left( \partial^\mu + iqA^\mu + i(\partial^\mu \alpha) - i(\partial^\mu \alpha) \right) \phi = e^{i\alpha} \left( \partial^\mu + iqA^\mu \right) \phi$$

Thus, the combination $(\partial^\mu + iqA^\mu)\phi$ transforms like $\phi$ itself! and the replacement $\partial^\mu \phi \to (\partial^\mu + iqA^\mu)\phi$ everywhere in the Lagrangian will make it invariant under local phase transformations of the field $\phi \to \phi' = e^{i\alpha}\phi$, and when we use the gauge freedom of the electromagnetic field to choose to simultaneously shift $A^\mu \to A'^\mu = A^\mu - \partial^\mu \alpha/q$. 

10/19/2009
The Covariant Derivative

The combination \((\partial^\mu + i q A^\mu)\) is defined as the **covariant derivative** denoted as \(D^\mu \equiv \partial^\mu + iqA^\mu\).

As example we show that the Klein-Gordon current in is indeed invariant with the replacement of \(\partial^\mu\) by \(D^\mu\):

\[
\begin{align*}
j'^\mu &= iq\left[\phi^*(D^\mu \phi'^\prime) - (D^\mu \phi'^\prime)\phi^*\right] = iq\left[e^{-i\alpha}\phi^\prime\left(e^{i\alpha}(D^\mu \phi) - (e^{-i\alpha}(D^\mu \phi^*))e^{i\alpha}\phi\right)\right] = \\
&= iq\left[\phi^*(D^\mu \phi) - (D^\mu \phi^*)\phi\right] = j^\mu
\end{align*}
\]

In summary: requiring local gauge invariance of the Lagrangian forces the introduction of a “counter term”, involving a vector **field** that exhibits the correct form of the (electromagnetic) interaction!

The electromagnetic field and its interactions is **generated** by the gauge principle, the freedom of choosing a phase locally for the particle fields.

This is very reminiscent of the theory of general relativity, where the form of the gravitational interaction follows directly from the freedom to choose my coordinate system locally, and no universal inertial system is required.
General Local Gauge Invariance

Often the local phase function \( \exp\{i\alpha(x)\} \) is written as \( \exp\{iq\alpha(x)\} \), which amounts to a redefinition of the space-time dependent function \( \alpha(x) \rightarrow q\alpha(x) \).

The parameter \( q \) can be thought of as the eigenvalue of the charge operator \( Q \), and, because the phase factor may be thought of as an operator itself, it can also be written as \( \exp\{iQ\alpha(x)\} \). This will become useful later.

We may generalize the local gauge transformations, and thus allow the gauge principle to prescribe the other types of interactions in nature: the weak and the strong interactions.

A general local phase transformation can be written as:

\[
\phi(x) \rightarrow \phi(x)' = U(x)\phi(x), \quad \text{with covariant derivative:} \quad D^\mu = \partial^\mu + igB^\mu
\]

\( U(x) \) must be a unitary transformation, because it should respect the normalization of the fields.
Transformation Properties of the Gauge Fields

A general local phase transformation can be written as:
\[ \phi(x) \rightarrow \phi(x)' = U(x)\phi(x), \]
with covariant derivative:
\[ D^\mu = \partial^\mu + igB^\mu \]
with \( U(x) \) a unitary transformation.

We can then derive the transformation property for \( B^\mu \) that allows the Lagrangian to be invariant when the covariant derivative replaces the “normal” derivative \( \partial^\mu \) everywhere:

\[ (D^\mu \phi)' = U(D^\mu \phi) \]

\[ \Rightarrow \quad (D^\mu \phi)' = D^\mu \phi' = (\partial^\mu + igB^\mu)'U \phi = (\partial^\mu + igB^\mu)U \phi = U(\partial^\mu + igB^\mu)\phi \]

\[ \Rightarrow \quad (\partial^\mu + igB^\mu)U \phi = U(\partial^\mu + igB^\mu)\phi \]

solve for \( B^\mu \):
\[ igB^\mu U \phi = -\partial^\mu (U \phi) + U(\partial^\mu \phi) + igUB^\mu \phi = -((\partial^\mu U)\phi + igUB^\mu \phi) = \left(-((\partial^\mu U) + igUB^\mu)\right)\phi \]

\[ \Rightarrow \quad B^\mu 'U = \left(\frac{i}{g} \partial^\mu U + UB^\mu\right) \quad \Rightarrow \quad B^\mu = UB^\mu U^{-1} + \frac{i}{g} (\partial^\mu U)U^{-1} \]

Note that we need to keep the order of operations because \( U \) may be a transformation matrix, for instance \( U(x) = \exp \{i \frac{1}{2} \vec{\sigma} \cdot \vec{b}(x)\} \), and \( B^\mu \) may be a vector or tensor.

To check, try \( U = e^{ia} \):
\[ A^\mu' = UA^\mu U^{-1} + i\frac{1}{q} (\partial^\mu U)U^{-1} \quad \text{with} \quad U = e^{ia}: \]
\[ A^\mu = e^{ia} A^\mu e^{-ia} + i\frac{1}{q} (\partial^\mu e^{ia})e^{-ia} = A^\mu + i\frac{1}{q} ie^{ia} (\partial^\mu \alpha)e^{-ia} = A^\mu - \frac{1}{q} (\partial^\mu \alpha) \]
**Consequences of Local Gauge Invariance**

Consider the Klein-Gordon equation for a bosonic field $\phi$, (e.g. the $\pi^+$); the complex conjugate field $\phi^*$ then represents the anti-$\pi^+$, i.e. the $\pi^−$.

We require $L$ to be invariant under local phase transformations of the $\pi^+$ field ($q=+e$), which **forces** the substitution $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + ieA^\mu$. Thus:

K-G equation \[
\left[\partial^2 + m^2\right]\phi \rightarrow \left[(\partial^\mu + ieA_\mu)(\partial^\mu + ieA^\mu) + m^2\right]\phi =
\]
\[
=\left[\partial^2 + ie(\partial^\mu A^\mu + A_\mu \partial^\mu) - e^2 A^2 + m^2\right]\phi \rightarrow \left[\partial^2 + m^2\right]\phi = -V \phi, \quad \text{with: } V \equiv ie(\partial^\mu A^\mu + A_\mu \partial^\mu) - e^2 A^2
\]

Where potential $V$ contains the interactions: a term of 1st order in $e$, and a 2nd order term. We will assume that $V$ vanishes at large distances from the “interaction region”.

The 1st order transition amplitude $T^{(1)}_{fi}$ from **initial state** $\phi_i$ to **final state** $\phi_f$ is then the integral:

$T^{(1)}_{fi} = -i \int d^4x \phi_f^*(x) V \phi_i(x) = +e \int d^4x \phi_f^*(x) (\partial^\mu A^\mu + A_\mu \partial^\mu) \phi_i(x) = +e \int d^4x \left[ \phi_f^* \partial^\mu (A^\mu \phi_i) + \phi_f^* A^\mu (\partial^\mu \phi_i) \right]$

$= +e \int d^4x \left[ -(\partial^\mu \phi_f^*) A^\mu \phi_i + \phi_f^* A^\mu (\partial^\mu \phi_i) \right] + \left[ e\phi_f^* A^\mu \phi_i \right]_S = -i \int d^4x \left[ \phi_f^* (\partial^\mu \phi_i) - (\partial^\mu \phi_f^*) \phi_i \right] A^\mu + 0$

$= -i \int d^4x j_{fi}^\mu A^\mu, \quad \text{with: } j_{fi}^\mu \equiv ie \left[ \phi_f^* (\partial^\mu \phi_i) - (\partial^\mu \phi_f^*) \phi_i \right] \]

The $\phi_i$ and $\phi_f$ are the **free particle** wavefunctions $ae^{−ipx}$, $a=(2E)^{−\frac{1}{2}}$, because they are defined (prepared or measured) far away from the interaction region both in time and in space.
The Current-Current Interaction

The interacting current can then be written as (with $\phi = a e^{-ipx}$):

$$j^f_\mu = i e \left[ \phi^*_f (\partial_\mu \phi_i) - (\partial_\mu \phi^*_f ) \phi_i \right] = i e a^*_f a_i \left[ -ip^i_\mu - ip^f_\mu \right] e^{ip^x_\mu - ip^x_\mu} = e \frac{1}{\sqrt{2E_f 2E_i}} \left[ p^f_\mu + p^i_\mu \right] e^{-i(p^f_\mu - p^i_\mu)x^\mu}.$$

The interaction can be thought of as a pion current $j_\mu$, which emits or absorbs a photon $A^\mu$.

The photon can be thought of as being “sunk” or “sourced” by second current, e.g. by a positively charged Kaon $K^+$.

In order to prevent (?) confusion we will number the participating particles – the pion and the Kaon – as follows: the incoming (initial state) particles are 1 (pion) and 2 (Kaon), and the outgoing particles are 3 (pion) and 4 (Kaon). The exchanged photon remains unnumbered.

The exchanged photon has a four-momentum equal to $q = p_3 - p_1$.

The pion and Kaon currents are then written (see equation ) as:

$$j^\mu_\pi = j^\mu_{13} = \frac{1}{\sqrt{2E_1 2E_3}} \left[ p_1 + p_3 \right]^\mu e^{-i(p_1 - p_3)x^\mu}, \quad j^\mu_K = j^\mu_{24} = \frac{1}{\sqrt{2E_2 2E_4}} \left[ p_2 + p_4 \right]^\mu e^{-i(p_2 - p_4)x^\mu},$$

with $p_1 - p_3 = -p_2 + p_4 = q$

Note that **Current Conservation requires**: $\partial_\mu j^\mu = 0$, thus: $q_\mu (p_1 + p_3)^\mu = q_\mu (p_2 + p_4)^\mu = 0$; this easily verified: $q_\mu (p_1 + p_3)^\mu = (p_1 - p_3)_\mu (p_1 + p_3)^\mu = m_1^2 - m_3^2 = 0!$
**The Current-Current Interaction**

The photon is related to its source/sink by the inhomogeneous Maxwell equations:  $\partial_\mu F^{\mu\nu} = j^\nu$.

We use the **Lorentz gauge-fixing** condition $\partial_\mu A^\mu = 0$ to obtain a simple equation for $A^\mu$:

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) = \partial^2 A^\nu = j^\nu$$  \textbf{(in the Lorentz gauge)}

Solving this 2\textsuperscript{nd} order differential equation for $A^\mu$:

$$\partial^2 A^\nu = j^\nu = j_{24} = \frac{1}{\sqrt{2E_2 E_4}} [p_2 + p_4]^\mu e^{-i(p_2-p_4)\cdot x} = \frac{1}{\sqrt{2E_2 E_4}} [p_2 + p_4]^\mu e^{iqx}$$

$$\Rightarrow \quad A^\mu = (iq)^{-2} \frac{1}{\sqrt{2E_2 E_4}} [p_2 + p_4]^\mu e^{iqx} = \frac{-1}{q^2} j_{24}^\mu$$

The 1\textsuperscript{st} order transition amplitude $T_{fi}^{(1)}$ becomes:

$$T_{fi}^{(1)} = -i \int d^4x j_{13} A^\mu = -i \int d^4x j_{\mu} 13 \frac{-1}{q^2} j_{24}^\mu = -i \int d^4x j_{13}^\nu \frac{-g_{\mu\nu}}{q^2} j_{24}^\mu =$$

$$= -i \frac{1}{\sqrt{2E_1 E_2 E_3 E_4}} e(p_1 + p_3)^\nu \frac{-g_{\mu\nu}}{q^2} e(p_2 + p_4)^\mu \int d^4x e^{-i(p_1+p_2-p_3-p_4)\cdot x} =$$

$$= -i \frac{1}{\sqrt{2E_1 E_2 E_3 E_4}} \int d^4x e^{-i(p_1+p_2-p_3-p_4)\cdot x} \times (-i) \left\{ i e(p_1 + p_3)^\nu \left( \frac{-i g_{\mu\nu}}{q^2} \right) i e(p_2 + p_4)^\mu \right\} =$$

$$= -i \prod_{j=1}^{4} \frac{1}{\sqrt{2E_j}} (2\pi)^4 \delta(p_1+p_2-p_3-p_4) \times (-i M_{fi}), \quad \text{with} \quad -i M_{fi} = i e(p_1 + p_3)^\nu \left( \frac{-i g_{\mu\nu}}{q^2} \right) i e(p_2 + p_4)^\mu$$
The Current-Current Interaction

\[ T_{fi}^{(1)} = -i \frac{1}{\sqrt{2E_1E_2E_3E_4}} \int d^4x e^{-i(p_1+p_2-p_3-p_4)x} \times (-i) \left( ie(p_1 + p_3)^\nu \left( -i \frac{g_{\mu\nu}}{q^2} \right) ie(p_2 + p_4)^\mu \right) \]

\[ = -i \prod_{j=1}^{4} \frac{1}{\sqrt{2E_j}} (2\pi)^4 \delta(p_1+p_2-p_3-p_4) \times (-i M_{fi}), \quad \text{with} \quad -i M_{fi} = ie(p_1 + p_3)^\nu \left( -i \frac{g_{\mu\nu}}{q^2} \right) ie(p_2 + p_4)^\mu \]

where the 1\textsuperscript{st} order single-boson-exchange matrix element \( M_{fi} \) has the universal form:

\[ -i M_{fi} = (\text{vertex factor})_{13} \times (\text{photon propagator}) \times (\text{vertex factor})_{24}. \]

The vertex factor differs for bosons and fermions, and the propagator differs for a gluon or for a weak vector boson, but the structure of the matrix element, and the “normalization” factors in front remain unchanged.

The delta distribution function represents energy-momentum conservation, and arises naturally from the integration of the exponentials in the free particle plane waves.

We have now derived the simplest Feynman rules for QED:

- a photon exchange gives rise to the factor \(-i g_{\mu\nu}/q^2\) (in the Lorentz gauge!)
- a scalar\(_1\)-boson-scalar\(_3\) vertex contributes a factor \(ie(p_1+p_3)^\mu\), where \(e\) is the charge and the momenta refer to the incoming and outgoing particles.

Using the Feynman rules, we can simply and efficiently calculate all graphs that contribute to a particular process.
Example of Toplogical Relations

For example, the muon decay diagram (a) is related to the exchange diagram (c) by a simple “flip” $p_2^\mu \rightarrow -p_2^\mu$ of particle 2’s momentum in the amplitude (!) and by the replacement of the particle by its backwards going antiparticle (note the arrows!):

$$q \equiv p_3 - p_1 = W$$
Example: Scattering of Spin-0 Particles: $a+b\rightarrow a+b$

We calculate the electromagnetic interaction of "spinless" $e^+$ and $e^-$ at 1st order in $e$:

These are expressed in terms of the matrix element $m_{fi}$.

This matrix element squared will be a Lorentz scalar, and we’ll express it in terms of Lorentz invariant quantities, e.g. in terms of the Mandelstam variables

$$s=(p_1+p_2)^2=m_1^2+m_2^2+2p_1p_2=(p_3+p_4)^2,$$

$$t=(p_1-p_3)^2=m_1^2+m_3^2-2p_1p_3=(p_2-p_4)^2,$$

and

$$u=(p_1-p_4)^2=m_1^2+m_4^2-2p_1p_4=(p_2-p_3)^2.$$

Similar expressions can be formed using energy-momentum conservation: $p_1+p_2=p_3+p_4$.

Spinless $e^-\mu^-\rightarrow e^-\mu^-; m_1=m_3, m_2=m_4$:

$$-im_{fi} = [ -ie(p_1+p_3)\nu ] \left( \frac{-ig_{\mu\nu}}{q^2} \right) [ -ie(p_2+p_4)^\mu ] = -i \frac{-e^2(p_1+p_3)_{\mu}(p_2+p_4)^{\mu}}{(p_1-p_3)^2}$$

This can be expressed simply in terms of $s$, $t$ and $u$:

$$-im_{fi} = i \frac{e^2}{q^2} (p_1+p_3)_{\mu}(p_2+p_4)^{\mu} = i \frac{e^2}{q^2} (p_1p_2+p_1p_4+p_3p_2+p_3p_4)$$

$m_1=m_3; m_2=m_4 \Rightarrow p_1p_2=p_3p_4; p_1p_4=p_2p_3$;

thus:

$$-im_{fi} = i \frac{e^2}{q^2} (2p_1p_2+2p_1p_4) = -i \frac{e^2}{t} (u-s)$$
Example: **Spinless** $e^- e^+ \rightarrow e^- e^+$

There are now **two** contributing diagrams:

1. a “$t$-channel” exchange diagram (as before)
2. an annihilation diagram or “$s$-channel” diagram

Both **amplitudes** must be added together!

We use $m_1=m_2=m_3=m_4$ in the last line:

$$-iM_{fi} = -i \frac{-e^2 (p_1 + p_3)_{\mu} (-p_2 - p_4)_{\mu}}{(p_1 - p_3)^2} +$$

$$-i \frac{-e^2 (p_1 - p_2)_{\mu} (-p_4 + p_3)_{\mu}}{(p_1 + p_2)^2}$$

$$= -i(-e^2) \left[ - \frac{S - U}{t} - \frac{T - U}{s} \right]$$
Example: **Spinless** $e^- e^- \rightarrow e^- e^-$

Again, there are two contributing diagrams:

Both are “$t$-channel” exchange diagrams; they differ because the final state particles (spinless bosons!) are indistinguishable.

Again, both amplitudes must be added together:

$$-iM_{fi} = -i \frac{-e^2 (p_1 + p_3)_\mu (p_2 + p_4)^\mu}{(p_1 - p_3)^2} +$$

$$-i \frac{-e^2 (p_1 + p_4)_\mu (p_2 + p_3)^\mu}{(p_1 - p_4)^2}$$

$$= -i(-e^2) \left[ \frac{s-u}{t} + \frac{s-t}{u} \right]$$

\[\begin{array}{c}
\text{Diagram 1:} \\
\text{Diagram 2:}
\end{array}\]
The Real Pion: Form Factors...

The real pion is built from quarks, and therefore will not have a point-like EM interaction. Although the current $j_\pi^\mu$ of the real pion will differ from the point-like “pion”, it must still be Lorentz contravariant vector!

At the scalar-photon-scalar vertex only 3 fourvectors are available: $p_1, p_3$, and $q=p_1-p_3$, of which only 2 are independent; choose: $q^\mu=(p_1-p_3)^\mu$ and $(p_1+p_3)^\mu$. Note that $q^\mu \perp (p_1+p_3)^\mu$.

Finally, there are only two Lorentz scalars at the vertex: $(p_1)^2 = (p_3)^2 = m_\pi^2$, and $q^2 = (p_1-p_3) = 2 m_\pi^2-2 p_1 p_3$ of which only one is independent, choose $q^2$.

The most general current fourvector that can be constructed for the real pion is then:

$$j_\pi^\mu = j_{13}^\mu = \frac{e}{\sqrt{2E_1 E_3}} \left[ F(q^2)(p_1 + p_3) + G(q^2)q \right]^\mu e^{-i(p_1-p_3)x}$$

Current conservation further restricts this to a single unknown Form Factor $F(q^2)$:

$$0 = \partial_\mu j_\pi^\mu = \partial_\mu j_{13}^\mu = \frac{e}{\sqrt{2E_1 E_3}} \left[ F(q^2)(p_1+p_3) + G(q^2)q \right]^\mu \partial_\mu e^{iq^\mu x} = \frac{e}{\sqrt{4E_1 E_3}} \left[ F(q^2)(p_1+p_3) + G(q^2)q \right]^\mu q_\mu e^{-iqx}$$

$$\Rightarrow F(q^2) q_\mu (p_1+p_3)^\mu + G(q^2)q^2 = 0 \Rightarrow G(q^2)=0 \text{ b.c.: } q_\mu (p_1+p_3)^\mu = (p_1-p_3)_\mu (p_1+p_3)^\mu = p_1^2 + p_3^2 = 0$$

Thus:

$$j_\pi^\mu = \frac{e}{\sqrt{2E_1 E_3}} F(q^2)(p_1 + p_3)^\mu e^{-iqx}$$

$F(q^2)$ is normalized to 1 at $q^2=0$, because $e$ is defined at large distance (zero momentum transfer $q$), i.e. at large distances the pion again appears point-like.
Massive Vector Bosons

Free massive spin-1 bosons are described by the free Klein-Gordon equation: \( \left( \partial^2 + M^2 \right) W^\mu = 0 \)

In the restframe of the \( W^\mu \) the spin can be characterized by a polarization 3-vector \( \varepsilon \), e.g.
expressed in the basis of linear polarization vectors \( \varepsilon_x=(1,0,0), \varepsilon_y=(0,1,0), \varepsilon_z=(0,0,1) \), or as circular and longitudinal polarization states \( \varepsilon_1=(\varepsilon_x + i\varepsilon_y)/\sqrt{2}, \varepsilon_2=(\varepsilon_x - i\varepsilon_y)/\sqrt{2}, \varepsilon_3=\varepsilon_z \).

Under Lorentz transformations, the polarization vector transforms as the 3-vector part of a fourvector \( \varepsilon^\mu=(0,\varepsilon) \) in the restframe. In the boson’s restframe we trivially have \( p_\mu \varepsilon^\mu=(M,0)^\mu g_{\mu\nu}(0,\varepsilon)^\nu=0 \), which, being a Lorentz scalar, will remain true in any frame.

Hence, a massive spin-1 object has only three independent polarization states, chosen, for instance, as 2 transverse and 1 longitudinal state. Note, that the notions transverse/longitudinal are not invariant!

The K-G equation for the massive vector field \( W^\mu \) above was first derived by Alexandru Proca (1938) from the the massless photon Lagrangian by the substitution \( \partial^2 \rightarrow \partial^2 + M^2 \):

\[
L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \rightarrow \quad L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} M^2 W_\mu W^\mu, \quad \text{with} \quad F^{\mu\nu} \equiv \partial^\mu W^\nu - \partial^\nu W^\mu
\]

The Euler-Lagrange equations of motion for \( W^\mu \) are:

\[
\left( \partial^2 + M^2 \right) W^\mu - \partial^\mu \partial_\nu W^\nu = \left\{ \left( \partial^2 + M^2 \right) g^{\mu\nu} - \partial^\mu \partial^\nu \right\} W_\nu = 0
\]
i.e. four independent equations.
Massive Vector Bosons

As an important topic in QED, we discuss **Vector Bosons**, i.e. bosons with spin-1, massive or not, that are carriers of the electromagnetic interaction.

Lagrange equation for a **massive** \( m=\) vector field \( W^\mu \):

\[
L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \rightarrow \quad L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} M^2 W_\mu W^\mu, \quad \text{with} \quad F^{\mu\nu} \equiv \partial_\mu W^\nu - \partial^\nu W_\mu
\]

The (four) Euler-Lagrange equations of motion for \( W^\mu \) are:

\[
\left( \partial^2 + M^2 \right) W^\mu - \partial_\mu \partial_\nu W^\nu = \left( \left( \partial^2 + M^2 \right) g^{\mu\nu} - \partial_\mu \partial_\nu \right) W_\nu = 0
\]

Acting on equation with \( \partial_\mu \) from the left yields \( M^2 \partial_\mu W^\mu = 0 \), i.e. \( \partial_\mu W^\mu = 0 \) if and only if \( M^2 \neq 0 \)!

So, if \( M^2 \neq 0 \) then \( \left( \partial^2 + M^2 \right) W^\mu = 0 \) is true as four independent equations with one constraint.

Note, that the constraint \( \partial_\mu W^\mu = 0 \) **directly results from the Lagrangian**, **not as a gauge choice**, but only so if \( M^2 \neq 0 \).

In 4-momentum space, the constraint corresponds to \( p_\mu \partial^\mu = 0 \) when we write the solution for the free field as \( W^\mu = \epsilon^\mu \epsilon^{-ipx} \).
The Propagator of the Massive Vector Boson

The propagator equation emerges when we add a point-source $\delta^4$ to the right-hand side of the free-boson equation:

$$\begin{array}{c}
\left\{ \left( \partial^2 + M^2 \right) g^{\mu\nu} - \partial^\mu \partial^\nu \right\} S_{\mu\nu} = \delta^4(x - x') \\
\end{array}$$

with formal solution:

$$S_{\mu\nu}(x, x') = \int d^4p \frac{e^{-ip(x-x')}}{(-p^2 + M^2) g^{\mu\nu} + p^\mu p^\nu}$$

The Fourier transform of $S(x, x')$ is $S(p) = \left\{ (-p^2 + M^2) g^{\mu\nu} + p^\mu p^\nu \right\}^{-1}$. This seems an “inverse” operator; let’s find its “normal” form:

$$\left\{ (-p^2 + M^2) g^{\mu\nu} + p^\mu p^\nu \right\} \left\{ A g_{\nu\rho} + B p_\nu p_\rho \right\} = \delta^\mu_\rho, \text{ with functions } A \text{ and } B \text{ to be determined...}$$

$$\Rightarrow \delta^\mu_\rho (-p^2 + M^2) A + (A + M^2 B) p^\mu p_\rho = \delta^\mu_\rho \Rightarrow (-p^2 + M^2) A = 1 \text{ and } (A + M^2 B) = 0$$

$$\Rightarrow A = \frac{1}{-p^2 + M^2}; \quad B = \frac{-1}{M^2(-p^2 + M^2)} \quad \Rightarrow S^{\mu\nu}(p) = -i \frac{-g^{\mu\nu} + p^\mu p^\nu/M^2}{p^2 - M^2}$$

$S(p)$ is then the massive vector boson propagator. When the propagator momentum is much smaller than the boson mass $M$:

$$S(q) \approx \frac{-ig^{\mu\nu}}{\left( q^2 - M^2 \right) q^2 \ll M^2} \approx \frac{ig^{\mu\nu}}{M^2}$$

Vector Boson Propagator
Massless Vector Bosons – the Photon

For the free photon with momentum $k^\mu$ no restframe exists and therefore the constraint $k_\mu \varepsilon^\mu = 0$ or $\partial_\mu A^\mu = 0$ is just a choice we make; other gauge choices are possible.

This Lorentz gauge $\partial_\mu A^\mu = 0$ is often chosen when dealing with tree-level diagrams, as here. The gauge choice reduces the 4 equations $\partial^2 A^\mu = 0$, to 3 independent ones.

For the free photon, this does not exhaust all freedom yet; i.e. we still have more freedom to choose $A^\mu$ without affecting the physical observables:

$$ B = \overrightarrow{\nabla} \times A \quad \text{and} \quad E = -\nabla^0 A - \overrightarrow{\nabla} A^0 \quad (E^i = F_{i0} = \overrightarrow{\nabla} A^0 - \nabla^0 A^i) $$

It is simple to check that any scalar field $\Lambda(x)$ with property $\partial^2 \Lambda = 0$ preserves both the Lorentz condition, as well as the physical fields $B$ and $E$.

This extra gauge freedom allows yet another constraint on the $\varepsilon^\mu$ components, which are therefore reduced to only 2 independent ones, typically chosen transverse:

choose $\Lambda = i \alpha \exp(-ikx)$

$$ \Rightarrow \partial^\mu \Lambda = \alpha k^\mu \exp(-ikx) \quad (\text{Note: } \partial^2 \Lambda = \alpha k^2 \exp(ikx) = 0 \text{ because } k^2 = 0) $$

then $A'^\mu = A^\mu + \partial^\mu \Lambda = (\varepsilon^\mu + \alpha k^\mu) \exp(-ikx) = \varepsilon'^\mu \exp(-ikx)$, with $\varepsilon'^\mu = \varepsilon^\mu + \alpha k^\mu$, 

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Polarizations of the Free Photon

choose $\Lambda = i\alpha \exp(-ikx)$

$$\Rightarrow \quad \partial^\mu \Lambda = \alpha k^\mu \exp(-ikx) \quad \text{(Note: } \partial^2 \Lambda = \alpha k^2 \exp(ikx) = 0 \text{ because } k^2 = 0)$$

then $A'^\mu = A^\mu + \partial^\mu \Lambda = (\varepsilon'^\mu + \alpha k^\mu) \exp(-ikx) = \varepsilon'^\mu \exp(-ikx)$, with $\varepsilon'^\mu = \varepsilon^\mu + \alpha k^\mu$.

The last expression, the arbitrary “shift” freedom, is an expression of the gauge freedom in QED;

it indicates that $\varepsilon^\mu$ and $\varepsilon'^\mu$, representing the same photon, are related by an arbitrary $\alpha$ times the photon’s four momentum, and thus we may always choose $\alpha$ such that $\varepsilon^0 = 0$

e.g. suppose that $\varepsilon^0 \neq 0$, then choose $\alpha = -\varepsilon^0/k^0$, then $\varepsilon^0 = \varepsilon^0 + \alpha k^0 = 0$.

Therefore, with this choice of gauge, the Coulomb gauge, the free photon is fully transverse: $\varepsilon^\mu k^\mu = \varepsilon \cdot k = 0$.

These two remaining independent components can be chosen as $\varepsilon_x$ and $\varepsilon_y$, or as left and right-circular polarizations.
Interacting Photons – Principle of Minimal EM Interaction

From the free-particle equations of motion \((\partial^2 + m^2)\psi = 0\) we went to the situation where photons interact with particles by requiring local gauge invariance of the free Lagrangian. This led to the “principle of minimal EM interaction”, which states that by making the replacement \(\partial^\mu \rightarrow \partial^\mu + ieA^\mu\) (with \(e\) the particle’s charge) we obtain the correct form of the coupling:

\[
0 = \left[ (\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) + m^2 \right] \psi = \left[ \partial_\mu \partial^\mu + ie \left( \partial_\mu A^\mu + A_\mu \partial^\mu \right) - e^2 A^2 + m^2 \right] \psi
\]

\[
\left( \partial^2 + m^2 \right) \psi = \left[ -ie \left( \partial_\mu A^\mu + A_\mu \partial^\mu \right) + e^2 A^2 \right] \psi
\]

The amplitude for emission of a photon \(A^\mu = e^\mu e^{-ikx}\) by a particle with momentum \(p_1\) before emission, and momentum \(p_2\) after emission, is:

\[
T_{21}^{(o)} = \int d^4x \psi_2^* V^{(o)}(x) \psi_1 = -ie \int d^4x \psi_2^* \left( \partial_\mu A^\mu + A_\mu \partial^\mu \right) \psi_1 = -ie \int d^4x \left[ -\psi_1 (\partial_\mu \psi_2^*) + \psi_2^* (\partial_\mu \psi_1) \right] A^\mu \\
= -i \left[ ie(p_1 + p_2)_\mu e^\mu \right] N_1 N_2 \int d^4x \exp \left[ -i(p_1 - p_2 + k)x \right] \\
= -i \left[ ie(p_1 + p_2)_\mu e^\mu \right] \frac{m^4}{m^4} N_1 N_2 2\pi^4 \delta^4(p_1 - p_2 + k),
\]

with the normalizations as before: \(N_j \equiv (2E_j)^{-1/2}\).
Interacting Photons

The interaction term quadratic in $e$ gives:

$$T_{21}^{(e^2)} = \int_{VT} d^4 x \psi_2^* V^{(e^2)} \psi_1 = e^2 \int_{VT} d^4 x \psi_2^* A^{(3)}_\mu A^{(4)\mu} \psi_1$$

$$= e^2 \epsilon^{(3)}_\mu \epsilon^{(4)\mu} N_1 N_2 N_3 N_4 \int_{VT} d^4 x \exp[-i(p_1-p_2+k_3+k_4)x]$$

$$= e^2 \epsilon^{(3)}_\mu \epsilon^{(4)\mu} N_1 N_2 N_3 N_4 (2\pi)^4 \delta^4(p_1-p_2+k_3+k_4)$$

and with proper symmetrization because 3 and 4 are indistinguishable:

$$T_{21}^{(e^2)} = -i \left[ ie^2 \left( \epsilon^{(3)}_\mu \epsilon^{(4)\mu} + \epsilon^{(4)}_\mu \epsilon^{(3)\mu} \right) \right]_{\mu} N_1 N_2 N_3 N_4 (2\pi)^4 \delta^4(p_1-p_2+k_3+k_4)$$
The Propagator of the Spin-0 Boson

In equation we derived the expression for the photon propagator from the inhomogeneous Maxwell equations (the Euler Lagrange equations of the EM Lagrangian).

We follow a similar argumentation in deriving the propagator of a massive spin-zero particle. The non-free Klein-Gordon equation is:

\[ (\partial^2 + m^2) \phi = J(x) \]

Again, we choose the source \( J(x) \) as a point source in space-time:

\[ (\partial^2 + m^2) \phi = \delta^4(x), \text{ which has formal solution } \phi(x) = \frac{-1}{(2\pi)^4} \int d^4 q \frac{e^{-iqx}}{q^2 - m^2}; \quad \phi(q) = \frac{-1}{q^2 - m^2} \]

check: \( (\partial^2 + m^2) \phi(x) = \frac{-1}{(2\pi)^4} \int d^4 q \frac{((-iq)^2 + m^2)e^{-iqx}}{q^2 - m^2} = \frac{+1}{(2\pi)^4} \int d^4 q e^{-iqx} = \delta^4(x) \)

where we identify the solution \( \phi(q) \), the Fourier transform of \( \phi(x) \), as the pion propagator.

Without proof: for consistency with higher-order loop-diagrams, a factor \(-i\) must be added to this propagator.
Example: Bremsstrahlung by a Scalar Boson

Bremsstrahlung occurs when an electron is accelerated in a nuclear Coulomb field and emits a photon, while Compton scattering is the process of X-ray photon scattering off quasi-free electrons in matter.

Both processes are pure QED and calculable (spinless!) with the tools we developed above. Take a spin-0 charged pion scattering in the Coulomb field of a heavy nucleus of charge $Ze$.

We count three diagrams; in the limit of very large nuclear mass $M$, we can ignore the diagrams in which the photon radiates off the “heavy” lines $p_1$ and $p_3$.

Note that the third diagram is really two because of the interchangeability/symmetrization of the two photons, i.e. there is an extra factor two in the matrix element for (3).

We simplify the formulae using transversality of the emitted real photon and assume the nucleus to be very massive (this also suppresses bremsstrahlung off $p_1$ and $p_3$!).
The matrix elements describing these Feynman diagrams are:

\[-iM_{fi}^{(1)} = i\epsilon(p_1 + p_3)_\mu \frac{-ig^{\mu\nu}}{q^2} i\epsilon(p_4 + p_4 - q)_\nu \frac{i}{(p_2 - k)^2 - m^2} i\epsilon(p_2 + p_2 - k)_\rho \epsilon^\rho = \frac{-2i\epsilon^3}{q^2} (p_1 + p_3)_\mu p_4^\mu \frac{2p_2 \cdot \epsilon}{-2p_2 \cdot k}\]

\[-iM_{fi}^{(2)} = i\epsilon(p_1 + p_3)_\mu \frac{-ig^{\mu\nu}}{q^2} i\epsilon(p_2 + p_2 + q)_\nu \frac{i}{(p_4 + k)^2 - m^2} i\epsilon(p_4 + k + p_4)_\rho \epsilon^\rho = \frac{-2i\epsilon^3}{q^2} (p_1 + p_3)_\mu p_2^\mu \frac{2p_4 \cdot \epsilon}{2p_4 \cdot k}\]

\[-iM_{fi}^{(3)} = i\epsilon(p_1 + p_3)_\mu \frac{-ig^{\mu\nu}}{q^2} 2\epsilon_\nu = \frac{+2i\epsilon^3}{q^2} (p_1 + p_3)_\mu \epsilon^\mu\]

where we used the following equalities:

\[k \cdot \epsilon = k_\mu \epsilon^\mu = 0, \quad (p_j \pm k)^2 - m^2 = m^2 \pm 2p_j \cdot k + 0 - m^2 = \pm 2p_j \cdot k, \quad j = 2, 4\]

\[(p_1 + p_3)_\mu q^\mu = (p_1 + p_3)_\mu (p_1 - p_3)^\mu = p_1^2 - p_3^2 = M^2 - M^2 = 0 \quad \text{(Current Conservation!)}\]

\[\Rightarrow (p_1 + p_3)_\mu (2p_j \pm q)^\mu = 2(p_1 + p_3)_\mu p_j^\mu, \quad j = 2, 4\]
Example: Bremsstrahlung by a Scalar Boson

The matrix elements describing the three Feynman diagrams are:

\[ -iM_{fi}^{(1)} = \frac{-2ie^3}{q^2} (p_1 + p_3)_{\mu} p_4^\mu \frac{2p_2 \cdot \epsilon}{-2p_2 \cdot k} \]

\[ -iM_{fi}^{(2)} = \frac{-2ie^3}{q^2} (p_1 + p_3)_{\mu} p_2^\mu \frac{2p_4 \cdot \epsilon}{2p_4 \cdot k} \]

\[ -iM_{fi}^{(3)} = \frac{+2ie^3}{q^2} (p_1 + p_3)_{\mu} \epsilon^\mu \]

Note, that the sum of all three terms is “gauge invariant”, i.e. it vanishes when \( \epsilon//k \): For instance, take \( \epsilon = \alpha k \), with \( \alpha \) some constant, then:

\[ \epsilon = \alpha k \quad \Rightarrow \quad (1) + (2) + (3) = \frac{2ie^3}{q^2} (p_1 + p_3)_{\mu} \left[ ap_4^\mu - ap_2^\mu + ak^\mu \right] \]

\[ = \frac{2ie^3}{q^2} a(p_1 + p_3)_{\mu} \left[ p_4^\mu - p_2^\mu + k^\mu \right] = \frac{2ie^3}{q^2} a(p_1 + p_3)_{\mu} (p_1 - p_3)^\mu = 0 \]

because of energy-momentum conservation.

Note that the sum is zero when \( \epsilon//k \), but not the individual terms separately. The gauge invariance principle applies to the full amplitude and so includes all orders in \( e \).

However, because \( e \) is an – a priori unknown – parameter, it is reasonable to assume that gauge invariance must hold order-by-order in \( e \).