Elementary Particle Physics
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Lecture 07

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Particle Kinematics – Part 3

Based on Special Relativity ...

In Particle Physics we use some special variables; e.g.:

• Rapidity \( \eta \equiv \frac{1}{2} \ln \left[ \frac{(E + p_z)}{(E - p_z)} \right] \)
• Center-of-Mass (\( \sum p_{in} = 0 \)) and Laboratory Systems (\( p_{\text{target}} = 0 \))

Notations:

• four-vector \( p \) or \( p^\mu \), (greek index \( \mu = 0,1,2,3 \))
• three-vector \( p \) or \( p^i \), (roman index \( i = 1,2,3 \))
Four Vectors and Special Relativity

A Lorentz transformation from an (unprimed) system to another (primed) system moving at relative velocity $\beta//+z$ with respect to the first, can be expressed in terms of speed $\beta$ and relativistic factor $\gamma ≡ (1−\beta^2)^{-\frac{1}{2}}$ as:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta \gamma & 0 & 0 & \gamma \end{pmatrix}, \text{ with } \det(\Lambda^\mu_\nu) = \gamma^2 - \beta^2 \gamma^2 = 1$$

Note the minus sign in front of $\beta \gamma$, when $\beta//+z$.

Also note the lower and upper indices: this facilitates the definition of a summation convention of same indices:

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu = \Lambda^0_0 x^0 + \Lambda^1_1 x^1 + \Lambda^2_2 x^2 + \Lambda^3_3 x^3 = \Lambda^0_0 t + \Lambda^1_1 x + \Lambda^2_2 y + \Lambda^3_3 z$$

This leads to a definition of a Lorentz-invariant four-dimensional dot-product:

$$x_\mu x^\mu = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2 = g_{\mu\nu} x^\nu x^\mu,$$

where the metric tensor $g_{\mu\nu}$ in special relativity is defined as: $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

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(Pseudo)Rapidity

The Lorentz transformation with the notation $\gamma = (1 - \beta^2)^{-1/2}$ can be viewed as a rotation in this (Minkowski) space using the rapidity variable $\eta$ (sometimes: “y”) defined as:
\[
cosh(\eta) \equiv \gamma; \quad \text{then } \sinh(\eta) = \beta \gamma \text{ and } \tanh(\eta) = \beta.
\]
The transformation matrix for a boost along $z$ becomes:
\[
\Lambda_{\mu, \nu} = \begin{pmatrix}
\gamma & 0 & 0 & -\beta \gamma \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\beta \gamma & 0 & 0 & \gamma \\
\end{pmatrix} = \begin{pmatrix}
cosh(\eta) & 0 & 0 & -\sinh(\eta) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh(\eta) & 0 & 0 & \cosh(\eta) \\
\end{pmatrix}
\]

Note that $e^{-\eta} = \frac{\sqrt{(1 - \beta)/(1 + \beta)}}{\sqrt{(E-p_z)/(E+p_z)}}$ \Rightarrow $\eta = \frac{1}{2} \ln \left[ \frac{(E+p_z)}{(E-p_z)} \right]

and $\eta = \text{arctanh}(\beta) = \text{arctanh}(p_z/E)$ for a boost along the $z$-axis!

In the case that the particle's mass is small compared to the energy $E$ one may write $p \approx E$ and $\eta \approx -\ln(\tan^{\theta/2})$, where $\theta$ is the polar angle between the direction of momentum $p$ and the $z$-axis (the boost-axis)

The variable $-\ln(\tan^{\theta/2}) \approx \eta$ is called the “pseudo-rapidity”.

Going between LABS $\Leftrightarrow$ CMS for a particle $i$: $\eta_i \Leftrightarrow \eta_i^* + \eta_{CMS}$
Proper and Improper Lorentz Transformations

Lorentz boosts are examples of the group of “proper” Lorentz transformations, i.e. transformations that have Det=+1 and are continuously linked to the unity transformation: for an infinitesimally small boost $\beta$ we have $\gamma = (1 - \beta^2)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\beta^2 \approx 1$ and $\beta \gamma \approx \beta$:

$$\Lambda_{\nu}^{\mu} (\beta \ll 1) = \delta_{\nu}^{\mu} + \delta \beta_{z} \begin{pmatrix} 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \end{pmatrix}_{\nu}$$

Similarly, a small angle rotation $\delta \phi$ around the z-axis can be written as:

$$R_{\nu}^{\mu} (\delta \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & \cos(\delta \phi) & \sin(\delta \phi) & 0 \\
0 & -\sin(\delta \phi) & \cos(\delta \phi) & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}_{\nu} \approx \delta_{\nu}^{\mu} + \delta \phi \lim_{\delta \phi \to 0} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}_{\nu}$$

which has Det($R$)=+1, another example of a proper Lorentz transformation and continuously connected to the unity transformation.

Any proper Lorentz transformation can be written as a succession of small boosts (along x, y, or z) and small rotations (around the x, y, or z-axis) – the generators of the Lorentz group.
Proper and Improper Lorentz Transformations

For example, a finite boost parallel to the \( z \)-axis with a boost parameter \( \eta \) can be constructed by a series of small boosts of size \( \delta \eta = \eta / n \) with \( n \to \infty \):

\[
\Lambda^\mu_\nu (\eta) = \lim_{n \to \infty} \left( \delta^\mu_\nu - \frac{\eta}{n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^n = \exp \left\{ -\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]

when expanding this exponential expression in a series, we find the result is identical to:

\[
\exp \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} =
\]

\[
= 1 + \left( \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} \right) + \frac{1}{2!} \left( \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} \right)^2 + \frac{1}{3!} \left( \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} \right)^3 + \ldots = 1 - \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\eta^2}{2!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\eta^3}{3!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\eta^4}{4!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \ldots
\]

\[
= \cosh \eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh \eta & 0 & 0 & -\sinh \eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix}
\]

where the “big 0” represents blocks of 3\( \times \)3 zeros. Note that for a small boost \( \eta \approx \delta \eta \) we have \( \cosh(\delta \eta) = \gamma \approx 1 \) and \( \sinh(\delta \eta) = \beta \gamma \approx \delta \eta \), i.e. the first two terms in the series expansion in...
Proper and Improper Lorentz Transformations

Similar expressions can be formed for rotations:

\[
\exp\left(\begin{pmatrix} 0 & +\phi \\ -\phi & 0 \end{pmatrix}\right) = \]

\[
= 1 + \begin{pmatrix} 0 & +\phi \\ -\phi & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & +\phi \\ -\phi & 0 \end{pmatrix}^2 + ... = 1 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{\phi^2}{2!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\phi^3}{3!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{\phi^4}{4!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\phi^5}{5!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - ...
\]

\[
= \cos \phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos \phi & \sin \phi & 0 \\ 0 & -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

The six generators of the proper Lorentz group are the matrices of the boosts and rotations.

Examples of improper transformations are the Time reversal and the Parity transformations which are \(\text{Diag}(-1,+1,+1,+1)\) and \(\text{Diag}(+1,-1,-1,-1)\), respectively. These improper transformations cannot be written as small departures from the unity transformation!
Lorentz Scalars

The length of a Lorentz fourvector is invariant under Lorentz transformations, as a special case of the invariance of the Lorentz fourvector dot product.

The four-momentum transfer is \( q \equiv p_3 - p_1 \), (\( q \) and \( p \) implied fourvectors). Note that the four-momentum transfer \( q \) is defined using external lines only; i.e. it is independent of the details of the interaction taking place in the volume.

The labels 1 and 3 are particle labels (Note labeling convention)

The Lorentz scalar \( q^2 \) is related to particles 1 and 3 only:

\[
q^2 = q_\mu q^\mu = (p_{3\mu} - p_{1\mu})(p_{3}^{\mu} - p_{1}^{\mu}) = p_3^2 + p_1^2 - 2p_{3\mu}p_{1}^{\mu}
\]

\[
= m_3^2 + m_1^2 - 2E_3E_1 + 2p_3 \cdot p_1 = m_3^2 + m_1^2 - 2E_3E_1 + 2|p_3||p_1|\cos \theta_{13}
\]

which is correct in any reference system. E.g. in the CMS \((p_1 + p_2 = 0)\)

and in the special case that \( m_1 = m_3 \), equation simplifies to:

\[
q^2 = m_3^2 + m_1^2 - 2E_3E_1 + 2|p_3||p_1|\cos \theta_{13}
\]

\[
m_1 = m_3 \rightarrow E_1 = E_3 = E, \ |p_1| = |p_3| = |p|
\]

\[
= 2m_3^2 - 2E^2 + 2p_2^2 \cos \theta_{13} = -2p_2^2(1 - \cos \theta_{13}) = -4p_2^2 \sin^2 \frac{\theta_{13}}{2}
\]

where \( p_2^2 \) is the three-momentum squared of the initial state particles.
Mandelstam Variables

Some often used generalized **Lorentz invariants** are the so-called Mandelstam variables \( s, t, \) and \( u. \)

They are defined as follows:

\[
s \equiv (p_1 + p_2)^2; \quad t \equiv (p_1 - p_3)^2 = q^2; \quad u \equiv (p_1 - p_4)^2;
\]

with property: \( s + t + u = \sum m_i^2 \)

Thus, these variables are not independent.

In the LABSYS, \( \sqrt{s} \) is equal to the length of the total initial four vector;

in the CMS \( \sqrt{s} \) simply equals the total initial energy:

\[
s = (p_1 + p_2)^2 = \left( \frac{E_1 + E_2}{p_1 + p_2 = p - p = 0} \right)^2 = (E_1 + E_2)^2
\]
Example: Producing an Anti-Proton

The lowest energy process for production of antiprotons from initial proton beam and target is $p\ p \rightarrow p\ p\ p\ p$.

The minimum energy for this reaction in the CMS is when all four final-state particles are produced at rest: $E_{\text{min}}^* = 4\ m_p$.

To find the minimum necessary Laboratory beam energy $E_1$ is simply found from the condition $s \geq 16\ m_p^2$. Then:

$$s = (p_1 + p_2)^2 = (E_1 + m_2)^2 - |p_1|^2 = 2m_2 E_1 + m_1^2 + m_2^2$$

$$E_1 = \frac{s - m_1^2 - m_2^2}{2m_2} \geq \frac{7}{m_1=m_2=m_p} \frac{7m_p}{s\geq(4m_p)^2}$$

at this energy protons are already quite relativistic: $\gamma = E_1/m_p = 7 >> 1$ and $\beta = p_1/E_1 = \sqrt{(E_1^2 - m_p^2)/E_1} \approx 1 - 1/2(m_p/E_1)^2 = 0.990$.

Note that while a LABSYS experiment requires a beam of 6.6 GeV, in the CMS one could do it with two colliding beams of 2.2 GeV. In other words: to produce a certain amount of mass-energy $\sqrt{s}$ in the CMS, the beam energy required in the LABSYS grows as $s$, whereas the required beam energy in a collider only grows like $\sqrt{s}$. 

10/12/2009
Cross Sections, Lifetimes, and Transition Amplitudes

From Transition Amplitude
to Cross Section and Life Time ...
Transition Matrix Element

Very generally, interactions can be viewed as the result of a “Scattering” operator $S$ acting on an initial state $|i\rangle$ leading to a multitude of possible final states.

The transition probability to a particular final state $|f\rangle$ is then given by the square of the $S$ matrix element $S_{fi} \equiv \langle f|S|i\rangle$.

Unitarity – the conservation of probability – requires that $S^\dagger S = 1$ for properly orthonormalized states:

$$\langle i|S^\dagger S|i\rangle = \sum_f \langle f|f\rangle = \langle i|i\rangle = 1 \quad \rightarrow \quad S^\dagger S = 1$$

i.e. the initial state $|i\rangle$ must result in some combination of final states $|f\rangle$ – which may include $|i\rangle$ itself, in which case no interaction took place.

To separate the no-interaction case from cases with interaction, one introduces a “Transition” operator $T$: $S \equiv 1+iT$, with elements $T_{fi} \equiv \langle f|S|i\rangle$, $f\neq i$.

It is further convenient, as will be shown below, to re-define $T$ in a way that makes energy-momentum conservation explicit:

$$T_{fi} \equiv (2\pi)^4 \delta^4(p_f - p_i)M_{fi}$$

The total transition probability, integrated over a space-time volume $V,T$, is then:

$$|T_{fi}|^2 = |\langle f|T|i\rangle|^2 = (2\pi)^8 \left[ \delta^4(p_f - p_i) \right]^2 |M_{fi}|^2$$
**Transition Probability**

The total transition probability, integrated over a space-time volume $V,T$, is then:

$$|T_{fi}|^2 = |\langle f | T | i \rangle|^2 = (2\pi)^8 \left[ \delta^4(p_f - p_i) \right]^2 |M_{fi}|^2$$

The squared $\delta$-function can be simplified by the following “trick”:

$$\left[ \delta^4(p) \right]^2 = \delta^4(p) \lim_{V,T \to \infty} \frac{1}{(2\pi)^4} \int d^4x \ e^{ipx} = \delta^4(p) \lim_{V,T \to \infty} \frac{VT}{(2\pi)^4}$$

thus:

$$|T_{fi}|^2 = (2\pi)^4 \delta^4(p_f - p_i) |M_{fi}|^2 \lim_{V,T \to \infty} VT$$

We are only interested in the **transition rate (i.e. transition probability per unit time) per unit volume**;

thus, as often in QM, we calculate the **transition probability per unit space-time volume**:

$$\omega_{fi} \equiv \frac{|T_{fi}|^2}{VT} = (2\pi)^4 \delta^4(p_f - p_i) |M_{fi}|^2 \left[ L^{-3}T^{-1} \right]$$
Cross Sections

Consider the process where two particles of masses $m_a$ and $m_b$ interact and produce $k$ particles in the final state:

$$|i\rangle: \ a + b \rightarrow |f\rangle: \ 1 + 2 + 3 + \cdots + k$$

Commonly, the cross section $\sigma$ for an interaction (which may be defined as loosely or specifically as desired) is pictured as the effective area of beam or target particle, such that when the two particles intersect within that area, the desired interaction occurs.

For a density $\rho_b$ (number per unit volume) of target particles $b$, the rate $R$ (number per unit time) of desired interactions equals the number of target particles contained in a cylinder of cross section $\sigma$ and length $|v_{rel}|$ (the relative speed of approach of the beam and target particles):

$$R = \rho_b \times \sigma \times |v_{rel}|.$$

If the beam contains $\rho_a$ particles of type $a$ per unit volume, the transition rate per unit volume

$$\omega = \rho_a \times R = \rho_a \rho_b \sigma |v_{rel}|.$$

Thus, we find the cross section for this process as:

$$d\sigma_{fi} = \frac{\omega_{fi}}{F_i} dN_f, \text{ with Flux } F_i \equiv \rho_a \rho_b |v_{rel}|; \quad [d\sigma] = \left[ L^{-3} T^{-1} \right] \div \left[ L^{-3} L^{-3} L T^{-1} \right] = \left[ L^2 \right].$$

The factor $dN_f$ is the number of possible final states that are considered in building $d\sigma_{fi}$, i.e. it is dimensionless ...
Cross Sections

\[ d\sigma_{fi} = \frac{\omega_{fi}}{F_i} dN_f, \text{ with Flux } F_i \equiv \rho_a \rho_b |v_{rel}|; \quad [d\sigma] = \left[ L^{-3} T^{-1} \right] \div \left[ L^{-3} L^{-3} L T^{-1} \right] = \left[ L^2 \right]. \]

The factor \( dN_f \) is the number of possible final states that are considered in building \( d\sigma_{fi} \), i.e. it is dimensionless.

In order to proceed, we again take the volume where the particles interact to be given by \( V \), and assume there to be one of each particle in the volume, i.e. \( \rho_a = \rho_b = 1/V \).

The number of final states \( dN_f \) is given by the phase space volume occupied by the final state particles: each particle \( j \) in \( V \), with \( \mathbf{p} \) between \( \mathbf{p}_j \leq \mathbf{p} \leq \mathbf{p}_j + d\mathbf{p}_j \), has a number density of states \( dn_j \) equal to the phase space volume \( V \cdot d^3p_j \) divided by the elementary volume \( h^3 \):

\[ dn_j = \frac{V \cdot d^3p_j}{h^3} = \frac{V}{(2\pi)^3} d^3p_j \quad \rightarrow \quad dN_f = \prod_{j=1}^{k} dn_j = \left( \frac{V}{(2\pi)^3} \right)^k \prod_{j=1}^{k} d^3p_j \]

As always, we’ll find ways to get rid of the factors \( V \) ...
Cross Sections

\[ d\sigma_{fi} = \frac{\omega_{fi}}{F_i} dN_f, \text{ with Flux } F_i \equiv \rho_a \rho_b |v_{rel}|; \quad [d\sigma] = \left[ L^{-3} T^{-1} \right] \div \left[ L^{-3} L^{-3} L T^{-1} \right] = \left[ L^2 \right]. \]

\[ dn_j = \frac{V \cdot d^3 p_j}{\hbar^3} = \frac{V}{(2\pi)^3} d^3 p_j \quad \rightarrow \quad dN_f = \prod_{j=1}^{k} dn_j = \left( \frac{V}{(2\pi)^3} \right)^k \prod_{j=1}^{k} d^3 p_j \]

The typical interaction only acts over a limited volume \( V \) and for a limited time, see Fig. 4.

Taking the interaction to occur at and around the space-time point \( x = (x,t) = 0 \), the particles in \( |i> \) and \( |f> \) will be described by free particle states in the limit \( x, t \to -\infty \) and \( x, t \to +\infty \) respectively:

i.e. \( \psi_j = N \exp(ip_j x_j) \), where \( N \) is a normalization such that \( \int \psi^* \psi d^3 x = N^2 V = 1 \), thus \( N^2 = 1/V \).

However, this normalization is not Lorentz invariant, and we must account for the length-contraction the interaction volume \( V \) undergoes when viewed from a system co-moving with the particle \( j \):

\[ N_j^2 = \frac{1}{\gamma_j V} \propto \frac{1}{2E_j V} \]

In the latter, the factor 2 is convention (one may think of this by considering that for a given energy \( E \) two different states exist with \( p \) and \( -p \)).

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Cross Sections

The normalization factors of all initial and final state particles are now “extracted” from the matrix element $M_{fi}$, leaving a ‘reduced’ matrix element $m_{fi}$:

$$
\left| M_{fi} \right|^2 = \left| m_{fi} \right|^2 \frac{1}{2E_a V} \frac{1}{2E_b V} \prod_{j=1}^{k} \frac{1}{2E_j V} = \left| m_{fi} \right|^2 \frac{1}{4E_a E_b V^2} \frac{1}{V} \prod_{j=1}^{k} \frac{1}{2E_j} \rightarrow
$$

$$
d\sigma_{fi} = \frac{(2\pi)^4 V^2}{|\mathbf{v}_{rel}|} \frac{V^k}{(2\pi)^3 k} \prod_{j=1}^{k} d^3 p_j \delta^4 \left( \sum_{j=1}^{k} p_j - p_a - p_b \right) \left| M_{fi} \right|^2 =
$$

$$
= \frac{(2\pi)^4}{4E_a E_b |\mathbf{v}_{rel}|} \prod_{j=1}^{k} \frac{1}{(2\pi)^3} \frac{d^3 p_j}{2E_j} \delta^4 \left( \sum_{j=1}^{k} p_j - p_a - p_b \right) \left| m_{fi} \right|^2
$$

Note that although this may seem complicated, it is actually simple: each final-state particle contributes a factor $d^3 p / [(2\pi)^3 2E]$ to the formula, a factor that is Lorentz invariant!

The flux factor is $4E_a E_b |\mathbf{v}_{rel}|$, also a Lorentz invariant under boosts along the direction of $\mathbf{v}_{rel}$.

The Lorentz invariant matrix element $m_{fi}$ will contain the **dynamics** of the particular problem, while the rest, in particular the Lorentz Invariant Phase Space factor $d\text{LIPS}$ contains the **kinematics**:

$$
d\text{LIPS}_{fi} = \prod_{j=1}^{k} \frac{d^3 p_j}{(2\pi)^3 2E_j} \delta^4 \left[ \sum_{j=1}^{k} p_j - p_i \right]
$$
Example: \( a + b \rightarrow 1 + 2 \) in CMS

Let’s take the simplest example with two particles in the final state. The cross section for the process \( a + b \rightarrow 1 + 2 \) is:

\[
\sigma(a + b \rightarrow 1 + 2) = \frac{(2\pi)^4}{4E_a E_b} \frac{1}{(2\pi)^6} \int \cdots \int \frac{dp_1}{2E_1} \frac{dp_2}{2E_2} \delta^4(p_1 + p_2 - p_a - p_b) \left| m_{a+b\rightarrow 1+2} \right|^2
\]

In the center-of-mass system (CMS) this form simplifies significantly:

In the CMS we have (by definition) \( p_a = -p_b = p_i \); thus \( p_1 = -p_2 = p_f \) and \( E_a + E_b = \sqrt{s} = E_1 + E_2 \) because of the \( \delta^4 \)-function (for simplicity of notation I leave out the \( * \)s that indicate the CMS quantities).

We use the four \( \delta \)-functions to eliminate four of the six integrations:

We will use:

\[
\delta (f(x)) = \delta (x) \sqrt{ \left[ \frac{\partial f(x)}{\partial x} \right]_{x=0} },
\]

and we'll also use:

\[
\frac{\partial E}{\partial p} = \frac{\partial}{\partial p} \sqrt{p^2 + m^2} = \frac{p}{E}
\]

and:

\[
|v_{rel}| = |\beta_a - \beta_b| = \left| \frac{p_i}{E_a} + \frac{p_i}{E_b} \right| = \left| \frac{p_i (E_a + E_b)}{E_a E_b} \right| = \left| \frac{p_i \sqrt{s}}{E_a E_b} \right|
\]

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Example: \( a + b \rightarrow 1 + 2 \) in CMS

\[
\delta(f(x)) = \delta(x) \left[ \frac{\partial f(x)}{\partial x} \right]_{x=0}, \quad \frac{\partial E}{\partial p} = \frac{\partial}{\partial p} \sqrt{p^2 + m^2} = \frac{p}{E}; \quad |v_{rel}| = |\beta_a - \beta_b| = \frac{|p_i|}{E_a} + \frac{|p_i|}{E_b} = \frac{|p_i|}{E_a E_b} = \frac{|p_i|}{E_a E_b}
\]

Then:

\[
\sigma(a+b\rightarrow l+2) = \frac{(2\pi)^4}{4E_a E_b |v_{rel}|} \int \cdots \frac{dp_1}{2E_1} \frac{dp_2}{2E_2} \delta^3(p_1 + p_2) \delta(E_1 + E_2 - \sqrt{s}) |m_{a+b\rightarrow l+2}|^2 = \frac{1}{16\pi^2 p_i \sqrt{s}} \int \left( \frac{p^2 dp}{4E_1 E_2} \right) \delta(E_1 + E_2 - \sqrt{s}) |m|^2 = \frac{1}{16\pi^2 p_i \sqrt{s}} \int \left( \frac{p^2 dp}{4E_1 E_2} \right) \delta(E_1 + E_2 - \sqrt{s}) |m|^2
\]

\[
= \frac{1}{64\pi^2 p_i \sqrt{s}} \int \left( \frac{p^2 dp}{E_1 E_2} \right) \left[ \frac{p + p}{E_1 + E_2} \right]^{-1} \delta(p - p_f) |m|^2 = \frac{1}{64\pi^2 p_i \sqrt{s}} \frac{p_f}{E_1 + E_2} \int d\Omega |m|^2
\]

\[
\Rightarrow \frac{d\sigma(a+b\rightarrow 1+2)}{d\Omega^*} = \frac{1}{64\pi^2 s} |p_f| |m_{a+b\rightarrow 1+2}|^2
\]

If the matrix element is a simple constant, \( |m_{fi}|^2 = 1 \), the integration over stereo angle \( \Omega \) is trivial: \( \sigma(a+b\rightarrow l+2) = (16\pi s)^{-1} |p_f/p_i| \), of dimension \( \text{GeV}^{-2} \). (apply \((\hbar c)^2\) for area in barn)
Example: $a + b \rightarrow 1 + 2$ in CMS

The momenta $|\mathbf{p}_i|$ and $|\mathbf{p}_f|$ of the initial and final state particles are fully determined by the total CMS energy and the particles' masses:

$$s = (E_1 + E_2)^2 = E_1^2 + E_2^2 + 2E_1E_2 = 2p^2 + m_1^2 + m_2^2 + 2\sqrt{(p^2 + m_1^2)(p^2 + m_2^2)}$$

thus:

$$(p^2 + m_1^2)(p^2 + m_2^2) = \frac{1}{4} \left( s - 2p^2 - (m_1^2 + m_2^2) \right)^2$$

$$p^* + \frac{p^2}{2} (m_1^2 + m_2^2) + m_1^2 m_2^2 = \frac{1}{4} \left( s^2 + 4p^4 + (m_1^2 + m_2^2)^2 - 4sp^2 - 2s(m_1^2 + m_2^2) + 4p^2 (m_1^2 + m_2^2) \right)$$

solve for $p$: $p^2 = \frac{1}{4s} \left( s^2 + (m_1^2 + m_2^2)^2 - 2s(m_1^2 + m_2^2) - 4m_1^2 m_2^2 \right) = \frac{1}{4s} \left( s^2 + (m_1^2 - m_2^2)^2 - 2s(m_1^2 + m_2^2) \right)$

$$= \frac{1}{4s} \left( s - (m_1 + m_2)^2 \right) \left( s - (m_1 - m_2)^2 \right) \equiv \frac{1}{4s} \lambda(s,m_1^2,m_2^2) \quad \rightarrow \quad |\mathbf{p}_f|^*_{CMS} = \frac{\sqrt[4]{\lambda(s,m_1^2,m_2^2)}}{2\sqrt{s}}$$

and similarly, $|\mathbf{p}_i|^* = \sqrt[4]{\lambda(s,m_a^2,m_b^2)/(2\sqrt{s})}$, with $\lambda(s,m_a^2,m_b^2)$ the so-called “triangle” function. Note that the triangle function is a Lorentz scalar!
Laboratory System $a + b \rightarrow 1 + 2$

Often the cross section is needed expressed in Laboratory system variables, e.g. for electron-proton scattering where the proton – the target $b$ – is at rest, i.e. $p_b = (m_b, 0)$.

In the CMS: $\lambda(s,m_a^2,m_b^2) = 4s|p_i,_{CMS}|^2$.

Expressed in Laboratory quantities ($b$ at rest) it becomes:

$$\lambda(s,m_a^2,m_b^2) = \left(s - (m_a + m_b)^2\right)\left(s - (m_a - m_b)^2\right) = s^2 + (m_a^2 - m_b^2)^2 - 2s(m_a^2 + m_b^2)$$

$$= (m_a^2 + m_b^2 + 2p_a p_b)^2 + (m_a^2 - m_b^2)^2 - 2(m_a^2 + m_b^2 + 2p_a p_b)(m_a^2 + m_b^2)$$

$$= 4(p_a p_b)^2 + 4(p_a p_b)(m_a^2 + m_b^2) + (m_a^2 + m_b^2)^2 + (m_a^2 - m_b^2)^2 - 2(m_a^2 + m_b^2 + 2p_a p_b)(m_a^2 + m_b^2)$$

$$= 4(p_a p_b)^2 + 2(m_a^4 + m_b^4) - 2(m_a^2 + m_b^2)^2 = 4\left[(p_{a \mu} p_{b;\mu})^2 - m_a^2 m_b^2 \right]$$

\[\downarrow\] In laboratory:

$$= (2E_a m_b - 2m_a m_b)(2E_a m_b + 2m_a m_b) = 4m_b^2(E_a^2 - m_a^2) = 4m_b^2 p_a^2|_{LABSYS}$$

In order to find the elastic scattering cross section in the lab, we first express the differential cross section in Lorentz invariant quantities, and the scattering angle in terms of $q^2 = t$:  

10/12/2009
Laboratory System $a + b \rightarrow 1 + 2$

In order to find the elastic scattering cross section in the lab, we first express the differential cross section in Lorentz invariant quantities, and the scattering angle in terms of $q^2 = t$:

At the $b,2$ vertex:
$$-2q \cdot p_b = -2(p_2 - p_b) \cdot p_b = 2m_b^2 - 2p_2 \cdot p_b = q^2$$

At the $a,1$ vertex:
$$q^2 \equiv (p_a - p_1)^2 = m_a^2 + m_1^2 - 2E_a E_1 + 2|p_a| |p_1| \cos \theta$$

solving for $E_1$:
$$q^2 = -2(E_a - E_1)m_b = -4E_a E_1 \sin^2 \frac{\theta}{2} \rightarrow E_1 = \frac{E_a}{1 + \frac{2E_a \sin^2 \frac{\theta}{2}}{m_b}}$$

i.e. for elastic scattering $E_1$ depends only on $\theta$!
**IN-elastic Scattering ...**

For inelastic scattering, where \( m_b \neq m_2 \), it is a function of two independent variables,
- e.g. \( E_1 \) and \( \theta \),
- or \( E_1 - E_a \) and the dimensionless variable “Bjorken-x”: \( x \equiv -2(p_b \cdot q)/q^2 \); for elastic scattering \( x = 1 \), else \( 0 \leq x < 1 \).

It is now straightforward to express the LABSYS and CMS angles in terms of the appropriate variables:

\[
\frac{d\sigma(a+b \rightarrow 1+2)}{d\Omega^*} = \frac{1}{64\pi^2 s} \left| \frac{p^*_f}{p^*_i} \right| \left| m_{a+b \rightarrow 1+2} \right|^2,
\]

\[
\frac{d\sigma(a+b \rightarrow 1+2)}{d\Omega} = \frac{1}{64\pi^2 m_b^2} \left( \frac{E_1}{E_a} \right) \left| m_{a+b \rightarrow 1+2} \right|^2
\]
Particle Decays

In case of decays, we are interested in the decay rate, i.e. the transition probability per unit time, see , integrated over all space:

\[ d\Gamma_{fi} \equiv \frac{|T_{fi}|^2}{T} dN_f = \lim_{V \to \infty} (2\pi)^4 \delta^4 (p_f - p_i) V |M_{fi}|^2 dN_f \quad \text{[T}^{-1}] \]

\[ = \lim_{V \to \infty} (2\pi)^4 \delta^4 (p_f - p_i) V \prod_{j=1}^{k} \left( \frac{V \cdot d^3 p_j}{(2\pi)^3} \right) |M_{fi}|^2 \]

\[ = \lim_{V \to \infty} (2\pi)^4 \delta^4 (p_f - p_i) V \left( \frac{1}{2E_a V} \prod_{j=1}^{k} \frac{1}{2E_j V} \right) \prod_{j=1}^{k} \left( \frac{V \cdot d^3 p_j}{(2\pi)^3} \right) |m_{fi}|^2 = \]

\[ = \frac{(2\pi)^4}{2E_a} \frac{1}{(2\pi)^3} \prod_{j=1}^{k} \left( \frac{d^3 p_j}{2E_j} \right) \delta^4 \left( \sum_{j=1}^{k} p_j - p_a \right) |m_{fi}|^2 \]
Example: \( a \to 1 + 2 \)

Again, a simple example is illustrative.

Consider the decay of a particle \( a \) into two final state particles 1 and 2 in the CMS of particle \( a \), where \( p_1 + p_2 = 0 \) and \( \sqrt{s} = E_a = m_a \):

\[
\Gamma(a \to 1 + 2) = \frac{(2\pi)^4}{2m_a} \frac{1}{(2\pi)^6} \int \ldots \int \frac{d\mathbf{p}_1}{2E_1} \frac{d\mathbf{p}_2}{2E_2} \delta^3(p_1 + p_2) \delta(E_1 + E_2 - m_a) \left| m_{a \to 1+2} \right|^2 = \\
= \frac{1}{8\pi^2 m_a} \iint \frac{dp_f}{4E_1 E_2} \delta(E_1 + E_2 - m_a) \left| m \right|^2 = \frac{1}{8\pi^2 m_a} \iint \frac{p^2 dp d\Omega}{4E_1 E_2} \left[ \frac{\partial(E_1 + E_2 - m_a)}{\partial p} \right]^{-1} \delta(p - p_f) \left| m \right|^2 = \\
= \frac{1}{32\pi^2 m_a} \iint \frac{p^2 dp d\Omega}{E_1 E_2} \left[ \frac{p}{E_1} + \frac{p}{E_2} \right]^{-1} \delta(p - p_f) \left| m \right|^2 = \frac{1}{32\pi^2 m_a} \frac{p_f}{E_1 + E_2} \iint d\Omega \left| m \right|^2 \\
\Gamma(a \to 1 + 2) = \frac{|p_f^*|}{32\pi^2 m_a^2} \iint d\Omega^* \left| m_{a \to 1+2} \right|^2 = \frac{\sqrt{\lambda(m_a^2, m_1^2, m_2^2)}}{64\pi^2 m_a^3} \iint d\Omega^* \left| m_{a \to 1+2} \right|^2
\]

For a unity matrix element the decay width is simply \( \Gamma(a \to 1 + 2) = |p_f^*|/(8\pi m_a^2) \), with dimension GeV\(^{-1}\); i.e. the matrix element must have dimension GeV.
Notes on the $\delta$-Function

Definition: \[ \int_{x_i}^{x_f} dx \ f(x) \ \delta(x-a) \equiv \begin{cases} f(a) & \text{if } a \in \{x_1, x_2\} \\ 0 & \text{if } a \notin \{x_1, x_2\} \end{cases} \]

Thus: \[ \delta(x) = \delta(-x) \]
\[ x \delta(x) = 0 \]

\[ \delta(f(x)) = \left( \sum_j \left. \left| \frac{\partial f(x)}{\partial x} \right| \right|_{x=x_j} \right)^{-1} \delta(x-x_j), \quad \text{where } x_j \text{ are the real roots of } f(x) = 0, \]

e.g.: \[ \delta(ax) = \frac{1}{|a|} \delta(x); \quad \text{and: } \delta(x^2) = \left| \frac{1}{2x} \right| \delta(x) \]

Many “representations” of the delta function exist. Some frequently used representations are:

1. the exponential function: \[ \int_{-\infty}^{+\infty} \frac{e^{ipx}}{(2\pi)^4} d^4x = \delta^4(p) \quad \text{(dimension GeV}^{-4}!) \]

2. The (normalized) **Breit-Wigner function** $\text{BW}(s;M,\Gamma)$ in the narrow-width approximation ($\Gamma \ll M$):
\[ \lim_{\Gamma \to 0} \left( \text{BW} \right) = \lim_{\Gamma \to 0} \frac{M\Gamma}{\pi} \frac{1}{(s-M^2)^2 + M^2\Gamma^2} = \delta(s-M^2) \quad \text{(dimension GeV}^{-2}!) \]
The Field Theories

**Local Gauge (Phase) Invariance** is a powerful guiding principle for building theories and finding the form of the interactions

Local Gauge Invariance is **necessary for renormalizability** of the Theory.

All theories realized in nature are believed to be locally gauge invariant theories.
Gauge Theories

All true present-day theories are based on **local gauge (or phase) invariance**, i.e. the *phase* of the wavefunctions can be varied according to an *arbitrary space-time dependent function*.

The Lagrangian $L = L(\psi, \partial_\mu \psi)$ that describes the system keeps the same form under phase transformations of the type:

$$\psi(r, t) \rightarrow \psi'(r, t) = e^{i\alpha(r, t)} \psi(r, t)$$

i.e. if $\psi(r,t)$ is a solution, then $\psi'(r,t)$ is a solution too!

This phase transformation is called “**local**”; the phase depends on the local coordinates $(r, t)$.

Transformations with constant phase (i.e. the same for all space-time points) are “**global**”.

The term “gauge” is inherited from attempts by Hermann Weyl to construct theories with invariance under local *scale* ("eich") transformations as a basis for electromagnetism.

**Local Gauge Invariance** is a powerful guiding principle, and is necessary for renormalizability of the calculations. All theories realized in nature are locally gauge invariant theories.

One can construct more complicated versions of the phase rotation, e.g. $\exp\{i\sigma \cdot b(x)\}$ which is a two-by-two matrix, as can be seen from the series expansion of the exp function:

$$e^{i\sigma \cdot b} = 1 + i\sigma \cdot b + \frac{(i\sigma \cdot b)^2}{2!} + \frac{(i\sigma \cdot b)^3}{3!} + \ldots$$

with $\sigma \cdot b = \sigma_1 b_1 + \sigma_2 b_2 + \sigma_3 b_3 = \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

with the standard representation of the Pauli matrices.
Quantum Electro-Dynamics

Electromagnetism is our most successful and most precisely tested theory. Maxwell’s equations have been tested from astronomical scales down to the sub-atomic domain.

In its description for particle physics, we start with the Maxwell Lagrangian:

\[ L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu, \]

with \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) the antisymmetric electromagnetic tensor. with the electric and magnetic fields defined in terms of \( A^\mu \equiv (V,A) \), scalar potential \( V \) and vector potential \( A \):

\[ B \equiv \vec{\nabla} \times A \quad \text{and} \quad E \equiv -\vec{\nabla} V - \frac{\partial}{\partial t} A \]

The **homogenous** Maxwell equations are satisfied identically by these definitions:

- \( \nabla \cdot B = \nabla \cdot (\nabla \times A) = 0 \) by the curl-grad theorem;
  this can be considered the *definition* of the vector potential \( A \): if \( \nabla \cdot B = 0 \), then \( B \) can be written as the curl of some vector field!
- \( \nabla \times E + dB/dt = -\nabla \times \nabla V - \nabla \times dA/dt + \nabla \times dA/dt = -\nabla \times \nabla V = 0 \) by the curl-grad theorem.
Quantum Electro-Dynamics: QED

With $A^\mu \equiv (V, \mathbf{A})$, the electromagnetic tensor $F^{\mu\nu}$, as defined, is composed of the electric and magnetic field components:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad \text{e.g.}$$

$$F^{0i} = \partial^0 A^i + \nabla_i V = -E^i = -E_i, \quad F^{ij} = \partial^i A^j - \partial^j A^i \quad \Rightarrow$$

$$F^{12} = -\frac{\partial}{\partial x} A_y + \frac{\partial}{\partial y} A_x = -B_z, \quad F^{13} = -\frac{\partial}{\partial x} A_z + \frac{\partial}{\partial z} A_x = +B_y, \quad F^{23} = -\frac{\partial}{\partial y} A_z + \frac{\partial}{\partial z} A_y = -B_x$$

$F^{\mu\nu}$ transforms like a Lorentz tensor under Lorentz transformations $\Lambda^\mu_\nu$.

This is shown using its definition, combined with the fact that the Lorentz transformation itself is only a function of the relative velocity of the two reference frames and independent of the space-time coordinates:

$$F'^{\mu'\nu'} = \partial'^\mu A'^\nu - \partial'^\nu A'^\mu = \Lambda^\mu_\lambda \partial^\lambda \Lambda^\nu_\rho A^\rho - \Lambda^\nu_\rho \partial^\rho \Lambda^\mu_\lambda A^\lambda = \Lambda^\mu_\lambda \Lambda^\nu_\rho \partial^\lambda A^\rho - \Lambda^\nu_\rho \Lambda^\mu_\lambda \partial^\rho A^\lambda = \Lambda^\mu_\lambda \Lambda^\nu_\rho F^{\lambda\rho}$$

which is the definition of a Lorentz-tensor transformation.
The **inhomogeneous** Maxwell equations are:

\[
\nabla \cdot E = \rho, \quad \nabla \times \mathbf{B} - \frac{d}{dt} \mathbf{E} = \mathbf{j}
\]

They are expressed compactly using \( F^{\mu\nu} \):

\[
j^\nu \equiv (j^0, \mathbf{j}) = \partial_\mu F^{\mu\nu} \equiv \partial_\mu \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) = \partial_\mu \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}
\]

The EM Lagrangian (density) is well known:

\[
L_{EM} = T - V = \frac{1}{2} \left( E^2 - B^2 \right) - \rho V + \mathbf{j} \cdot \mathbf{A} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu
\]

The first term is the kinetic energy term,

The last term contains the potential energy, i.e. the forces and interactions between the fourvector field (the photon) and the electric sources (currents and charges).

The first term is the only remaining term in the absence of sources of the electromagnetic field, and describes the propagation of free photons in vacuum.

We’ll show that the **interactions**, represented by the second term, will be a **consequence of the requirement of local gauge invariance**. Later, we will introduce interactions by requiring the Lagrangian to be invariant under local gauge/phase transformations of its charged particle fields that serve as sources for the electromagnetic field!
The \textit{inhomogeneous} Maxwell equations are the Euler-Lagrange equations of the $L_{\text{EM}}$:

Hamilton’s principle states that the true “action” – the four-integral of the Lagrangian density – must be at a minimum, i.e. invariant under small changes in the generalized coordinates $A^\nu$ and velocities $\partial^\mu A^\nu$:

$$0 = \frac{\partial L}{\partial A^\nu} - \partial^\mu \left( \frac{\partial L}{\partial (\partial^\mu A^\nu)} \right) = -j^\nu + \frac{1}{4} \partial^\mu \left( g_{\lambda\sigma} g_{\rho\tau} \frac{\partial (F^{\sigma\tau} F^{\lambda\rho})}{\partial (\partial^\mu A^\nu)} \right) =$$

$$= -j^\nu + \frac{1}{4} \partial^\mu \left( g_{\lambda\mu} g_{\rho\nu} F^{\lambda\rho} - g_{\lambda\nu} g_{\rho\mu} F^{\lambda\rho} + F^{\sigma\tau} g_{\mu\sigma} g_{\nu\tau} - F^{\sigma\tau} g_{\nu\sigma} g_{\mu\tau} \right) =$$

$$= -j^\nu + \frac{1}{4} \partial^\mu \left( F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \right) = -j^\nu + \partial^\mu F_{\mu\nu}, \quad \Rightarrow \quad \partial_{\mu} F^{\mu\nu} = j^\nu$$

These are indeed the inhomogeneous Maxwell equations: e.g. for $\nu=0$ we obtain:

$$j^0 = \rho = \partial_{\mu} F^{\mu0} = \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = 0 + \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = \bar{\nabla} \cdot \mathbf{E}$$

and for $\nu = 1 = x$ we get:

$$J^1 = j_x = \partial_{\mu} F^{\mu x} = \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = -\frac{\partial}{\partial t} E_x + 0 + \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y = -\left( \frac{\partial \mathbf{E}}{\partial t} \right)_x + \left( \bar{\nabla} \times \mathbf{B} \right)_x$$

\textbf{Current conservation} follows automatically: $\partial_{\nu} j^\nu = \partial_{\nu} (\partial_{\mu} F^{\mu\nu}) = 0$, using the definition of $F^{\mu\nu}$. 

10/12/2009
Gauge Freedom of $A^\mu$

The fourvector field $A^\mu$ is not fully determined by the physically observables $E$ and $B$. We have the freedom to vary $A^\mu$ by the addition of a gradient of a scalar function/field: $E$ and $B$ remain unchanged under a “shift”:

$$A^\mu \rightarrow A^\mu'(x) = A^\mu(x) + \partial^\mu \chi(x).$$

As proof it suffices to observe that the electromagnetic field tensor $F^{\mu\nu}$ (which has all the $E$ and $B$ field components as its elements) is invariant under such shifts of $A^\mu$:

$$F^{\mu\nu} \rightarrow F^{\mu\nu'} = \partial^\mu A^{\nu'} - \partial^{\nu} A^\mu' = \partial^\mu A^\nu + \partial^\mu \partial^\nu \chi - \partial^\nu A^\mu - \partial^\nu \partial^\mu \chi = \partial^\mu A^\nu - \partial^\mu A^\nu = F^{\mu\nu}.$$  

This extra freedom represents the fact that a local change in the electrostatic potential $V$ is compensated by a local change in the magnetic vector potential $A$ (‘t Hooft, 1980).

Varying the field $A^\mu$, while keeping the four-current fixed, and requiring the EM Lagrangian of to be invariant:

$$0 = \delta \int d^4x \mathcal{L} = \int d^4x \left[ L(A^\nu', \partial^\mu A^\nu') - L(A^\nu, \partial^\mu A^\nu) \right] = -\int d^4x j_\nu \partial^\nu \chi = \int d^4x \left( \partial^\nu j_\nu \right) \chi \quad \Rightarrow \quad \partial^\nu j_\nu = 0$$

i.e. current conservation is required by, and follows from, local gauge invariance (invariance under “shifts” of the fourvector potential)! 

Expressed in the fourvector potential, the Maxwell equations read:

$$j^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$$

10/12/2009
Gauge Fixing: the Lorentz Gauge and the Coulomb Gauge

Gauge invariance allows us to restrict the form of $A^\mu$, called “gauge-fixing”, i.e. imposing a particular condition on the $A^\mu$ (corresponding to a particular choice for $\chi$).

Common gauge-fixing conditions are the **Radiation or Coulomb gauge** condition: $\nabla \cdot A = 0$; and the **Lorentz gauge** condition: $\partial_\mu A^\mu = 0$.

Fixing the gauge is often useful to simplify calculations or to expose particular features of the theory. In the “Lorentz gauge” the Maxwell equations simplify to $j^\nu = \partial_\mu \partial^\mu A^\nu = (\partial_t^2 - \nabla^2) A^\nu$.

The **Coulomb gauge-fixing** condition is clearly *not* Lorentz invariant. The Maxwell equations in the Coulomb gauge become: $j^\nu = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_0 A^0$.

For the charge distribution ($\nu=0$) this is $\rho = j^0 = \partial_\mu \partial^\mu A^0 - \partial^0 \partial_\mu A^\mu = \partial_\mu \partial^i A^0 = -\nabla^2 V$, with the Coulomb solution:

$$V(r, t) = A^0 = \frac{1}{4\pi} \int d^3 r' \frac{\rho(r', t)}{|r - r'|}$$

The three-current satisfies the equation:

$$j^i = j^i = \partial_\mu \partial^\mu A^i - \partial^i \partial_\mu A^\mu = -\frac{\partial^2 A}{\partial t^2} - \nabla^2 A + \frac{\partial}{\partial t} \nabla V$$

Thus, $V$ is solely determined by the charge distribution and nothing else; it has no independent dynamics.

The Coulomb gauge fixing condition $\nabla \cdot A = 0$ tells us that the three components of $A$ are interdependent: of the *four* fields $A^\mu$, there remain only *two physically independent* fields!
**Massless Vector Bosons**

In the absence of sources and currents \( j^\mu = 0 \) the Coulomb potential \( V \) is zero (or constant), and equation

\[
\mathbf{j} = j^i = \partial_\mu A^\mu A^i - \partial^i \partial_0 A^0 = \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \frac{\partial}{\partial t} \nabla V
\]

Becomes the wave equation for free photons: \( \nabla^2 \mathbf{A} = \partial^2 \mathbf{A} / \partial t^2 \).

With solutions: \( \mathbf{A}(\mathbf{r},t) = a \mathbf{\epsilon} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \), with amplitude \( a \) and polarization vector \( \mathbf{\epsilon} \).

For the free photon the Coulomb gauge condition \( \nabla \cdot \mathbf{A} = 0 \) translates into \( \mathbf{k} \cdot \mathbf{\epsilon} = 0 \) (or \( \mathbf{p} \cdot \mathbf{\epsilon} = 0 \) with \( \hbar \)), i.e. a free photon has only two independent components transverse to its direction!

The fourvector field \( A^\mu \) represents the photon, the carrier of electromagnetic interactions. \( A^\mu \) is a four-vector field: **the photon is a spin-1 (vector) boson**.

All force carriers are vector bosons (except for the spin-2 Graviton). In preparation for the discussion of massive vector bosons \( W^\pm (80 \text{ GeV}) \) and \( Z^0 (91 \text{ GeV}) \), we consider the addition of a mass term to the Lagrangian:

\[
L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu
\]

A gauge transformation of \( A^\mu \rightarrow A^\mu'(x) = A^\mu(x) + \partial^\mu \chi(x) \) is clearly not leaving this Lagrangian invariant: new terms like \( A_\mu \partial^\mu \chi \) and \( \partial^\mu \chi \partial_\mu \chi \) are introduced by \( A^\mu \rightarrow A^\mu'(x) = A^\mu(x) + \partial^\mu \chi(x) \).

Thus, **gauge invariance requires the free photon to be massless**.

This is fine for electromagnetism, but is a problem for the weak vector bosons: if they also have to stay massless, then the theory is definitely not going to describe reality!