Beyond The Standard Model

The Standard Model has been extraordinarily successful in describing the phenomenology of particle interactions at a very large range of energies, from muon decay and the $g-2$ values for electron and muon, to very high energy quark and gluon interactions at the Fermilab Tevatron proton-antiproton collider. All this was accomplished with a modest number of “free parameters”. QCD-based Lattice Gauge calculations are starting to successfully describe the particle mass spectrum with the “bare” quark masses as only inputs.

Altogether, the Standard Model as it stands contains at least 9 (6 “bare” quark and 3 lepton masses) + 5 ($U(1)$, $SU(2)$, $SU(3)$ couplings and two Higgs potential parameters) + 4 (Quark mixing matrix) = 18 arbitrary parameters. The large number of “arbitrary” parameters points to the fact that the Standard Model is incomplete and must be based on a more fundamental theory; it is only a “low-energy” effective theory. Even so, the Standard Model relates many, on first glance very disparate, processes, and it is therefore exquisitely testable. Hundreds of “tests”, i.e. single measurements of masses, lifetimes, cross sections; as well as measured distributions have given results that are in exquisite agreement with predictions, within the experimental and theoretical uncertainties.

9.1 Running Coupling Constants: QED

Up to now we have only considered “tree” diagrams, i.e. diagrams that are first-order (in the coupling constants). Of course, numerous more complicated Feynman diagrams need to be considered for a complete computation of the cross sections. Often, the higher-order diagrams are contributing significantly to the cross section or even overwhelm the tree-level result. In other cases, a certain process can only be described by a second- or higher-order diagram. In the section below, we discuss – mostly qualitatively – the computation of higher-order diagrams, and the effects they have on the physical coupling constants and masses. Consider the tree-level diagram in Figure 25. The matrix element is, see (III.103):

$$-iM_{ji}^{(1)} = \left( i e u^\gamma u_1 \right) \left( -i g_\mu^\nu \frac{q^2}{q^2} \right) \left( i u_3^\gamma u_2 \right)$$  \hspace{1cm} (III.211)

Similarly, we can calculate the second-order diagram in Figure 25. This is now more complex, because of the fermion loop in the propagator:

$$-iM_{ji}^{(2)} = \left( i e u^\gamma u_1 \right) \left( -i g_\mu^\nu \frac{q^2}{q^2} \right) \int_0^\infty \frac{d^4 p}{(2\pi)^4} \left\{ i e^\gamma \right\}_{\alpha \beta} \left( i e^\gamma \right)_{\alpha \beta} \frac{i \left( p' + m_f \right)}{p^2 - m_f^2} \frac{i \left( p - m_f \right)}{(q - p)^2 - m_f^2} \left( i e^\gamma \right)_{\gamma \delta} \left( i e^\gamma \right)_{\gamma \delta}$$

$$= \left( i e u^\gamma u_1 \right) \left( -i g_\mu^\nu \frac{q^2}{q^2} \right) I^{\sigma \alpha} \left( i u_3^\gamma u_2 \right), \text{ with } I^{\sigma \alpha}(q^2) = \int_0^\infty \frac{d^4 p}{(2\pi)^4} \text{ Tr} \left\{ i e^\gamma \right\}_{\alpha \beta} \frac{i \left( p' + m_f \right)}{p^2 - m_f^2} \frac{i \left( p - m_f \right)}{(q - p)^2 - m_f^2} \left( i e^\gamma \right)_{\gamma \delta} \left( i e^\gamma \right)_{\gamma \delta}$$  \hspace{1cm} (III.212)
The term in the square brackets, defined as $I_{\rho \sigma}$, is new. The implied summation over loop fermion spins together with the fact that the virtual photon spin components must match, is indicated by the spin indices $\alpha - \delta$. $I_{\rho \sigma}$ appears to be a quadratically divergent integral: $d^4 p = |p|^2 dp$ while the trace contains a power $|p|^{-2}$, making the integral infinite. We will circumvent this for now by replacing the upper limit with a (large) mass $M$, and later take the limit $M \to \infty$. 

$$I_{\rho \sigma}(M; q^2)$$ can be evaluated:

$$\begin{align*}
I_{\rho \sigma}(M; q^2) &\equiv \frac{M^2}{(2\pi)^2} \text{Tr} \left[ (ie\gamma^\rho) \left( \frac{p^\rho + m_f}{p^2 - m_f^2} \right) \left( e\gamma^\sigma \right) \left( \frac{q - p}{(q - p)^2 - m_f^2} \right) \right] \\
&= -ig_{\rho \sigma} q^2 \left[ \frac{e^2}{12\pi^2} \int \frac{dp^2}{p^2} - \frac{1}{2\pi^2} \int_0^1 dz z (1-z) \ln \left( 1 - \frac{q^2 z (1-z)}{m_f^2} \right) \right] \\
&= -ig_{\rho \sigma} q^2 \left[ \frac{e^2}{12\pi^2} \ln \frac{M^2}{m_f^2} - \frac{1}{2\pi^2} \int_0^1 dz z (1-z) \ln \left( 1 - \frac{q^2 z (1-z)}{m_f^2} \right) \right] \\
&\equiv -ig_{\rho \sigma} q^2 I(M; q^2)
\end{align*}$$

The first term in the square brackets contains the infinity. The second term is finite; for small $q^2$, respectively large $q^2$ it equals:

$$\begin{align*}
\lim_{-q^2 \to 0} \frac{e^2}{2\pi^2} \int_0^1 dz z (1-z) \ln \left( 1 - \frac{q^2 z (1-z)}{m_f^2} \right) &= \frac{e^2}{2\pi^2} \frac{-q^2}{m_f^2} \int_0^1 dz z^2 (1-z)^2 = \frac{e^2}{2\pi^2} \frac{1}{30} = \frac{e^2}{60\pi^2} m_f^2 \\
\lim_{-q^2 \to \infty} \frac{e^2}{2\pi^2} \int_0^1 dz z (1-z) \ln \left( 1 - \frac{q^2 z (1-z)}{m_f^2} \right) &= \frac{e^2}{2\pi^2} \ln \left( \frac{-q^2}{m_f^2} \right) \int_0^1 dz z (1-z) = \frac{e^2}{12\pi^2} \ln \left( \frac{-q^2}{m_f^2} \right)
\end{align*}$$

The large $q^2$ approximation leads to:

$$I(M; q^2) = \frac{e^2}{12\pi^2} \ln \frac{M^2}{-q^2}$$  \hspace{1cm} (III.215)

The extraction of the factor $q^2$ allows us to consider the second diagram as a correction to the propagator of the tree-level diagram:

$$\begin{align*}
-ig_{\mu \nu} &\to -ig_{\mu \nu} + \left( \frac{-ig_{\mu \nu}}{q^2} \right) I_{\mu \sigma} \left( \frac{-ig_{\nu \sigma}}{q^2} \right) = -ig_{\mu \nu} - ig_{\mu \nu} q^2 \frac{-I(M; q^2)}{q^2} q^2 I(M; q^2) = -ig_{\mu \nu} \frac{1 - I(M; q^2)}{q^2} \\
&\approx -ig_{\mu \nu} \left( 1 - \frac{e^2}{12\pi^2} \ln \frac{M^2}{m_f^2} - \frac{e^2}{60\pi^2} m_f^2 \right) q^2 I(M; q^2)
\end{align*}$$  \hspace{1cm} (III.216)

where we used the small $q^2$ approximation. We see the second-order propagator correction contains a (logarithmically) infinite term, as well as a finite correction that depends on $q^2$ (note: $q^2 < 0$). Higher order terms come in (e.g. two successive loops, three successive loops, etc.).

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which lead to $O(e^4)$ and higher corrections... We may choose to absorb the total correction in the coupling constant $e$, producing an effective coupling constant $e$:

$$e^2 = e_0^2 \left(1 - I(M;q^2)\right) \quad \rightarrow \quad e^2(q^2) = e_0^2 \left(1 - I(M;q^2) + (I(M;q^2))^2 - (I(M;q^2))^3 + \ldots\right) = e_0^2 \left(\frac{1}{1 + I(M;q^2)}\right)$$

(III.217)

where $e_0$ is some sort of “bare” or “primordial” coupling. Indeed, the effective coupling is truly what the experiments measure, and clearly this must be finite. The physical coupling “constant” $e_{\text{eff}}$ thus depends on $q^2$: $e = e(q^2)$ We can measure $e$ at some “scale” $q^2 = -\mu^2$, and relate the measured $e(Q^2)$ to the $e$ measured at another scale $q^2 = -\mu^2$. For large $q^2$:

$$e^2(\mu^2) = \frac{e_0^2}{1 + I(M; -\mu^2)} \approx \frac{e_0^2}{1 + \frac{e_0^2}{12\pi^2} \ln \frac{M^2}{\mu^2}}$$

$$\Rightarrow \quad e^2 = e_0^2 \left[1 + \frac{e_0^2}{12\pi^2} \ln \frac{M^2}{\mu^2}\right]$$

(III.218)

Thus, in the comparison, the infinity is removed! The physical quantities remain finite and the infinities is “hidden” in the unmeasurable (infinitesimal) bare coupling constant $e_0$. The coupling constant now “runs” (increases) with the $Q^2$ of the measurement. The running is extremely small in most situations. However, the effect is well observable and has been measured by comparing $\alpha$ measured in atomic physics, $\alpha = 1/137.035999679(94)$, with the measurement of $\alpha$ at the Z-mass ($m_Z=91.1876$ GeV): $\alpha = 1/127.925(16)$. The running is actually increased, hence the variable $n_f$ in the last expression of (III.218), because additional fermion species can be “excited” at large $Q^2$: e.g. quark-antiquark pairs, muon pairs, etc.

Finally, it is of interest to note that the result (III.218) is a special (and a large $q^2$ approximated) result that emerges from solving the so-called Renormalization Group Equation for QED:

$$Q \frac{\partial g(Q)}{\partial (\ln Q)} = \beta(g(Q)), \quad \text{with} \quad \beta(g) = \frac{g^3}{16\pi^2} b_0 + \frac{g^5}{(16\pi^2)^2} b_1 + \ldots$$

for QED: $g^{\text{QED}} = e$ and $b_0^{\text{QED}} = \frac{4}{3} \left(3 \sum_f Q_f^2 + \sum_l Q_l^2\right)$

(III.219)

where the $\beta$-function is the characteristic function of the particular gauge theory group, in this case QED. The $b_i$ increase when more leptons $l$ or quark flavors $f$ with charges $Q_f$ and $Q_l$ enter the picture. The factor $[12\pi^2]^{-1}$ in (III.218) is, in fact, the first term of $\beta(g) = b_0/(16\pi^2)$ in the case of a single $ee$ loop when $b_0 = 4/3$.

9.1.1 Higher-Order Vertex Corrections

We have considered second-order corrections due to loops in the propagator; but what about loops around the vertex? Several second-order diagrams need consideration, shown in Figure 26. These diagrams have been calculated in detail, and similar to the propagator correction, they lead to a correction of the current. A full calculation shows that, as before, both infinite and finite correction terms emerge. The finite terms, in the small $q^2$ approximation are:
\[ i e u_i \gamma^\mu u_i \rightarrow i e u_i \left[ \gamma^\mu \left\{ 1 + \frac{e^2}{12\pi^2} \frac{q^2}{m_f^2} \left( \ln \frac{m_f}{m_i} - \frac{3}{8} \right) \right\} - \frac{e^2}{8\pi^2} \frac{1}{2m_f} i \sigma_{\mu\nu} q^\nu \right] u_i \]  

(III.220)

where the \( m_i \) is an ad-hoc cut-off mass for the photon in the photon-fermion loops, which can be cancelled by careful calculation of the emission of very-soft (“infra-red”) photons that are contained in the overall cross section because the detector will, for instance, miss soft photons below its detection threshold. The finite term multiplying \( \gamma^\mu \) is again absorbed into a redefinition of the electric charge \( e \). The second term is a “magnetic” term, similar in form to the one we encountered in the Pauli equation in (III.92). This term leads to a deviation of the gyromagnetic ratio \( g_e \) away from 2:

\[ -\frac{e}{2m_e} i \sigma_{\mu\nu} q^\nu \Rightarrow \mu = -\frac{e}{2m_e} \sigma = 2 -\frac{e}{2m_e} S = g_e^{(1)} -\frac{e}{2m_e} \rightarrow \mu = \left( 1 + \frac{e^2}{8\pi^2} \right) -\frac{e}{2m_e} S = 2 \left( 1 + \frac{e^2}{8\pi^2} \right) -\frac{e}{2m_e} S = g_e^{(2)} -\frac{e}{2m_e} S \]  

(III.221)

i.e. to second order the \( g_e = 2 + \alpha/\pi \). Calculated to \( \alpha^4 \), the anomalous part of the gyromagnetic ratio is:

\[ \frac{g_e - 2}{2} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 - 0.328478965 \left( \frac{\alpha}{\pi} \right)^2 + 1.17562(56) \left( \frac{\alpha}{\pi} \right)^3 - 1.472(152) \left( \frac{\alpha}{\pi} \right)^4 + 4.46 \times 10^{-12} \]  

(III.222)

in excellent agreement with experiment.

The infinite terms from the vertex loops can be shown to exactly cancel in the sum of the three diagrams of Figure 26! This is true order-by-order! This is very good, because these infinite terms, being dependent on the mass of the particular fermion (electron, muon, etc.) interacting with the photon, once “absorbed” into an effective coupling, would entail different physical charges for the electron and muon! Note, that propagator loops did not present such a difficulty, because they are insensitive to the particle that scatters. Thus, only vacuum polarization – the loops in the propagator – modify the charge, and the running coupling is the same for all fermions with the same bare charge. The order-by-order cancellations of infinities in the vertex corrections are called Ward identities, and are a consequence of local gauge invariance in the theory. The absorption of infinities into an effective (running) coupling constant is called renormalization of the coupling.

### 9.2 Running of the Strong Coupling Constant

In the above, we used QED as the framework for the discussion of higher-order effects in the cou-
pling and the interaction. The treatment of QCD loop diagrams goes along the same lines. Because gluons are colored themselves, the loops are not only fermion loops but also gluon loops, and one for each type of gluon! Because of this, the running of the QCD coupling \( \alpha_s \), see Figure 27, is dramatically different from \( \alpha \). The strong coupling strength is the product of the color charge (i.e. the “normalization constant” of the gluon wavefunction, e.g. the factor \( 1/\sqrt{6} \) in the gluon \( 1/\sqrt{6}(\bar{RR} + \bar{GG} - 2\bar{BB}) \)) at the vertex with an exchanged gluon multiplied with \( \sqrt{\alpha_s/2} \); where the factor \( 1/2 \) is historical. This is similar to QED, where the coupling strength is \( \sqrt{Q^2} \).

In the running coupling constant expression for QED, equation (III.218), we must make the replacement – given without derivation:

\[
\frac{e^2(\mu^2)}{12\pi^2} = \frac{\alpha(\mu^2)}{3\pi} \rightarrow \frac{\alpha_s(\mu^2)/2}{3\pi} \left[ \frac{\ln f - 33/2}{12\pi} \right] = \frac{\alpha_s(\mu^2)}{12\pi} (2n_f - 33), \tag{III.223}
\]

where the first term between square brackets accounts for loops of all quark flavors energetically allowed, the second – opposite polarity(!) – term for all gluon loops. The large third term inverses the polarity of the running of \( \alpha_s \):

\[
\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 - \left(2n_f - 33\right)\frac{\alpha_s(\mu^2)}{12\pi} \ln \frac{Q^2}{\mu^2}} = \frac{\alpha_s(\mu^2)}{1 + \left(33 - 2n_f\right)\frac{\alpha_s(\mu^2)}{12\pi} \ln \frac{Q^2}{\mu^2}} \tag{III.224}
\]

Thus, in a world with less than 17 flavors, the coupling constant decreases to zero with increasing \( Q^2 \); QCD is asymptotically free! Instead of the arbitrary scale \( \mu^2 \) it is custom to pick a scale \( Q^2 = \Lambda^2 \) at which \( \alpha_s(Q^2) \) becomes large, i.e.:

\[
\left(33 - 2n_f\right)\alpha_s(\mu^2) \ln \frac{\Lambda^2}{\mu^2} = -1 \Rightarrow \frac{\Lambda^2}{\mu^2} = \exp \left\{ \frac{-12\pi}{\left(33 - 2n_f\right)\alpha_s(\mu^2)} \right\} \Rightarrow \alpha_s(Q^2) = \frac{12\pi}{\left(33 - 2n_f\right)\ln \frac{Q^2}{\Lambda^2}}, \tag{III.225}
\]

Using the measurements, one finds that \( \Lambda \approx 200 \text{ MeV} \), equivalent to a distance scale of 1 fm, i.e. the size of a hadron. Indeed, this is the scale at which we expect perturbative QCD to break down because of the “blow-up” of the strong coupling constant!

### 10  Grand Unified Theory and Supersymmetry

Taking the results from the previous section, one may plot the three SM couplings versus energy, see Figure 28. A precise calculation, taking account of all known quark and lepton flavors, yields the left hand graph, where the couplings unify, although not very well with present experimental precision. Introducing Supersymmetry, which postulates a fermion-boson symmetry and requires a scalar particle (sleptons, squarks) for every fermion and a spin \( \frac{1}{2} \) particle particle for every gauge boson (gaugino: wino, zino, gluino, …), allows us to “kink” the running in such a way as to bring the unification about precisely, see the right hand graph. This requires additional particles to change the slopes of the running couplings, with masses around \( Q = 10^{2.5} \sim 100-1000 \text{ GeV} \). If such particles are there, the couplings unify at a unification scale of \( Q = 3 \times 10^{16} \text{ GeV} \), still well below the Planck mass \( 1.22 \times 10^{19} \text{ GeV} \). This unification is not as trivial as it may seem because the slopes of the couplings are strongly correlated.

Now, the theory of Gravity, in four space-time dimensions, is not renormalizable, because Newton’s coupling constant \( G \) is dimensionful. The argument is very similar to the original four-point Fermi theory for the weak interaction, where the Fermi coupling \( G_F \) has likewise a dimension of \( \text{GeV}^{-2} \). In the latter case, we saw that the Fermi theory was only an effective theory at low energy, and that \( G_F \propto g^2/(q^2 - M_W^2) \).
This saves the weak theory until much higher energy, and the Higgs boson has to help out at even higher energy in preserving the renormalizability because in the end only theories possessing full local gauge invariance, and therefore massless gauge bosons, are renormalizable. The large amount of symmetry available in Yang-Mills theories is absolutely required for renormalizability; it was this proof by ‘t Hooft that made the GWS theory a mature and accepted “Standard Model” for particle physics!

However, leaving the search for a unification with Gravity to other graduate courses, there is a good case to be made for unification of $SU_C(3) \otimes SU_L(2) \otimes U_Y(1)$, called a “Grand Unification Theory” or GUT, at the “intermediate” scale of $M_G=3\times10^{16}$ GeV. The GUT group $G$ must then break down into the gauge sub-groups for the Color charge, the Lethanded weak iso-charge, and the Hypercharge $Y$ at energies below $M_G$ and the coupling strengths for these three charges evolve down in different ways because of their different particle content.

The Standard Model has the following particle content for each of three families (generations):

\[
Q = \begin{pmatrix}
  u^c_L, & \bar{u}^c_L, & d^c_L, & \bar{d}^c_L, & \nu^c_L, & \bar{\nu}^c_L, & e^c_L
\end{pmatrix}_L \quad \text{transfording as} \quad \begin{pmatrix}
  3, & 2, & 1, & 1, & 3, & 5, & 5
\end{pmatrix}
\]

plus their anti-particles. In this list we used all lefthanded states. The list totals 15 states in total, differing in color, 3rd component of weak isospin, and $Y$. In the SM we have $(3^2-1) + (2^2-1) + 1 = 12$ gauge (vector) bosons, plus a single Higgs $SU(2)$ doublet, which – after symmetry breaking – results in a single (scalar) Higgs boson. Each symmetry group has its own coupling: $g_3$, $g$, and $g'$ respectively, or equivalently: $\alpha_S=g_3^2/4\pi$, $\alpha=e^2/4\pi$ with $e=gsin\theta_W$, and $sin^2\theta_W=g^2/(g^2+g'^2)$. There are 15 more parameters: 4 CKM parameters and 9 fermion masses (assuming massless neutrinos), plus the two Higgs potential parameters $\mu$ and $\lambda$. In the case of massive neutrinos we must add at least 7 more parameters.

The smallest GUT group that can encompass $SU_C(3) \otimes SU_L(2) \otimes U_Y(1)$ is the special (Det=1) unitary group of rank 5 $SU(5)$, that can be generated by traceless unitary $5 \times 5$ matrices:

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & b_{14} & b_{15} \\
a_{12}^* & a_{22} & a_{23} & b_{24} & b_{25} \\
a_{13}^* & a_{23}^* & a_{33} & b_{34} & b_{35} \\
b_{14}^* & b_{24}^* & b_{34}^* & c_1 & c_{12} \\
b_{15}^* & b_{25}^* & b_{35}^* & c_{12} & c_{22}
\end{pmatrix}
\]

\[
\text{with } a_{11} + a_{22} + a_{33} + c_{11} + c_{22} = 0
\]
where the $3 \times 3$ $a$-part are the color generators $\lambda_{ij}$ of $SU(3)$, and the $2 \times 2$ $b$-part the weak iso-spin Pauli matrices of $SU(2)$, and the one traceless diagonal matrix is the Hypercharge operator of $U(1)$. The vectors that this “matrix” acts on are 5-component objects. The first choice will be the object $(q_R^r, q_R^g, q_R^b, \nu_L^r, e_L^r)$, i.e. a righthanded quark of the three colors $r, g, b$, and the lefthanded lepton doublet, a total of 5 states. In the language of group theory, this is a $\mathbf{5}$ representation: $\mathbf{5} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) = (d_L^r) + (L)$. The Hypercharge operator that commutes with all of the $SU(5)$ generators is, in the representation (III.227), the traceless matrix $\text{Diag}(Y_q, Y_q, Y_q, Y_e, Y_e)$, thus $3Y_q + 2Y_e = 0$. With $Y_e = 2(Q - I_W)e = -1$ and $I_q = 0$ (no weak coupling), this imposes $Y_q = \pm 2/3$, or $Q_q = \pm 1/3$, i.e. the lefthanded anti-down quark $d_L^r$.

The $b$-part couples quarks and leptons, i.e. “rotates” quarks into leptons and vice versa, and thus violates conservation of baryon number $B$ and lepton number $L$, but conserves $B - L$. In GUTs proton decay is natural and is indeed hard to avoid.

The remaining 10 fermions $(Q, \bar{u}, \bar{e})$ fit into a (antisymmetric tensor) representation: $\mathbf{10} = (\mathbf{3}, \mathbf{2}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}) = (Q, \bar{u}, \bar{e})$:

$$
\begin{pmatrix}
0 & \bar{u}_b & -\bar{u}_g & -\bar{u}_r & -d_r \\
0 & u_r & -u_g & -d_g \\
0 & -u_b & -d_b \\
0 & -\bar{e}
\end{pmatrix}
$$

(III.228)

$SU(5)$ has $5^2 - 1 = 24$ gauge bosons, i.e. 12 more than the SM. The new 12 gauge bosons, named $X$ and $Y$, must be very massive in order to suppress proton decay. The proton lifetime is measured to be larger than $10^{33}$ yrs by SuperKamiokande and Soudan II, and its decay via the exchange of a heavy $X$ or $Y$ gauge bosons can be estimated as:

$$
\Gamma \left( p \rightarrow \pi^0 + e^+ \right) \approx \frac{G_F^2 m_p^5}{192\pi^3} = \frac{g_s^4 m_p^5}{6144 \pi^3 M_X^4} = \frac{\alpha_s^2 m_p^5}{384 \pi M_X^4} \Rightarrow M_X \geq 10^{14} \text{ GeV} \quad \text{(III.229)}
$$

This is getting close to the value of $M_X$ for GUTs. In fact, without supersymmetry, $M_X$ is lower, and SU(5) GUT predicts too low a value for the proton lifetime!

### 10.1 Problems with the Standard Model

For all its success, there exist now several theoretical hints that the Standard Model is an effective theory which will break down at very high energy, and that a more fundamental theory must underlie the SM. In the sections below, we discuss several serious flaws in the Standard Model that lead us to such a conclusion.

#### 10.1.1 Cross Sections at Ultra-High Energy

Cross sections for several processes become very large at ultra-high energy. For instance, the cross section for $WW$ scattering at high energy “blows up”, i.e. violates unitarity. It grows like $s$. This is obvious for purely dimensional reasons, because of the presence of a dimension-full coupling in the form of $G_F^2$ in the cross section, with dimension GeV$^{-4}$, we must “compensate” by a factor $(hc)^2s$, see also (III.140). The high-energy behavior is improved, but not fully cured, by inclusion of the Higgs-exchange diagram.
**10.1.2 the Cosmological Constant or the Hierarchy Problem**

In the Standard Model, the vacuum expectation value is non-zero. For the Higgs potential as chosen: \(V(\phi) = -\frac{1}{2}\mu^2 \phi^2 + \frac{1}{4}\lambda^2 \phi^4\), the vacuum expectation value occurs at \(\phi_0 = |\mu/\lambda| \equiv v\), and has the value of \(V(\phi_0) = -\frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda^2 v^4\). Experimentally, we can calculate \(v\) (in the framework of the Standard Model): \(v = 246\) GeV, but \(\mu\) is only connected to the Higgs mass: \(m_{\text{Higgs}} = \mu\sqrt{2}\).

The Higgs mass is not predicted in the Standard Model, but – as argued in the previous section – it must be less than about a TeV. Experimentally, the lower mass limit is at 115 GeV (LEP2). Altogether, the Standard Model Higgs potential vacuum value is at least \(V(\phi_0) = -\frac{1}{4}\mu^2 v^2 \approx -10^8\) GeV\(^4\) = \(-10^8\) GeV\(^4\)/(8\(\times\)10\(^{-42}\) GeV\(^3\)\(\cdot\)cm\(^3\)) \(\approx -10^{49}\) GeV/cm\(^3\), i.e. 10\(^9\) times denser than a neutron star!

Experimentally, this is very far from the observed energy and matter density in our universe: visible matter accounts for about 1 proton per cubic meter. There are about 10\(^{10}\) soft photons for each proton, with a mean energy of about 3 Kelvin = 3\(\times\)10\(^{-4}\) eV, totaling about 3\(\times\)10\(^6\) eV = 3 GeV per cubic meter. Hence, the total energy density of the universe seems to be less than 10 GeV/m\(^3\) = 10\(^{-5}\) GeV/cm\(^3\). There is therefore a HUGE discrepancy between the Higgs vacuum energy and the observation.

However, Einstein’s theory for general relativity allows the insertion of an arbitrary constant in the mass potential, which doesn’t affect the Einstein equations. The possibility of this “cosmological constant” was introduced by Einstein, who himself never liked it very much. Such a constant term may account for the discrepancy between the Higgs potential and the observed energy density, but it has to be exceedingly (and unrealistically) finely tuned in order to so precisely cancel the Higgs potential from \(-10^{49}\) GeV/cm\(^3\) to \(10^{-5}\) GeV/cm\(^3\)! This is called the “fine-tuning problem” or the “problem of the cosmological constant”, and it is yet one more hint that the Standard Model is, to say the least, incomplete.

An equivalent statement is called the “Hierarchy Problem” in the SM. The GUT theory, whatever its details may be, require ultra-massive gauge fields and therefore a (set of) Higgs fields at the GUT energy scale around 10\(^{16}\) GeV. These GUT Higgs bosons contribute, just like any other fermions or bosons in the theory, to corrections to the “bare” mass of the SM Higgs boson, see Figure 29. These corrections are proportional to the mass of the bosons in the loop and are therefore divergent by themselves. Corrections from Fermion loops and Boson loops come in with a different sign. Therefore, if there are new scalar bosons at very high energy \(M_X\), they will produce a running of the SM Higgs mass as:

\[
m_{\text{Higgs}}^2(M_X) = m_{\text{Higgs}}^2(M_W) + g^2 C_1 (M_X^2 - M_W^2) + g^2 C_2 \ln \frac{M_X}{M_W} + O(g^4)
\]

(III.230)

where \(C_1\) is a dimensionless constant and \(g\) is the coupling constant. In order to avoid a physical SM Higgs mass larger than 1 TeV, there must be an extremely finely tuned difference of order 10\(^5\) GeV\(^2\) between two very large numbers of order 10\(^{32}\) GeV\(^2\): \(m_{\text{Higgs}}^2(M_X)\) and \(g^2 C_1 M_X^2\), which is very “unnatural”.

Figure 29. Loop corrections to the bare SM Higgs mass.