3.7 \textit{ep Scattering}

Scattering of electrons off protons has been an early tool for the detailed study of the electromagnetic structure of the proton. In the mid to late sixties, experiments at the electron linac at Stanford University provided first evidence of point-like objects inside the proton, experiments that were the relativistic analog of Rutherford scattering. Let us compare the expectation for scattering off a point-like object (the \(e\mu\) elastic scattering of before), with elastic scattering off an extended object like the proton. For \(e\mu\) scattering we found:

\[
\left| m_{e\mu \to e\mu} \right|^2 = \frac{8e^4}{q^4} \left\{ (p_1^\mu p_2^\nu + p_3^\mu p_4^\nu) - m_e^2 (p_2^\mu p_4^\nu - m_p^2 p_1^\mu p_3^\nu) + 2m_e^2 m_p^2 \right\} = 2e^4 \frac{s^2 + u^2}{t^2} \tag{III.121}
\]

In the CMS and the Laboratory system this leads to the differential cross section formulae:

\[
\frac{d\sigma_{e\mu \to e\mu}}{d\Omega} = \frac{1}{64\pi^2 s} \frac{1}{t^2} e^4 \frac{4\alpha^2 + 4\cos^4 \theta}{4s} \ , \quad d\sigma_{e\mu \to e\mu} \left|_{\text{CMS}} \right. = \frac{4\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \left( \cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2} \sin^2 \frac{\theta}{2} \right) \tag{III.122}
\]

For \(ep\) scattering the proton vertex presents a problem: because the proton is not a point particle, it will not be a simple solution to the Dirac equation. In order to proceed, we invoke general symmetry principles that limit the form that the proton current can take: 1) We invoke charge current conservation at the \(\gamma^\nu p\) vertex, and 2) we invoke general Lorentz invariance of the matrix element, which requires that the proton tensor \(K^\mu\nu\) transforms, like the lepton tensor \(L^\mu\nu\), as a proper Lorentz tensor under Lorentz transformations:

\[
\left| m_{ep \to ep} \right|^2 = \frac{e^4}{q^4} L^\mu\nu_{(e)} K_{(p)\mu\nu} \tag{III.123}
\]

From QED we found (using the Casimir trick and the trace algorithms) that:

\[
L^\mu\nu_{(e)} = 2 \left( p_\mu^\nu p_\nu^\rho + p_\mu^\rho p_\rho^\nu + g^\mu\nu (m_e^2 - p_1^\rho p_3^\rho) \right) = 2p_\mu^\nu p_\nu^\rho + 2p_\mu^\rho p_\rho^\nu + g^\mu\nu q^2
\]

is a Lorentz tensor and therefore \(K^\mu\nu\) must also be a Lorentz covariant tensor. It is symmetric in the indices 1 and 3. As a tensor describing the proton-photon vertex, it will exhibit its space-time properties through the properties of the fourvector variables of that vertex: \(p_2^\mu, p_4^\mu,\) and \(q^\mu.\) Of these three fourvectors only two are independent, and we choose \(p_2^\mu = p^\mu\) (the initial-state target fourvector) and \(q^\mu\) which is fully determined from the electron fourvectors: \(q^\mu = (p_1 - p_3)^\mu.\) Using these two variables, the most general Lorentz covariant form of the proton tensor is:

\[
K^\mu\nu_{(p)} = K_1 p^\mu p^\nu + K_2 g^\mu\nu q^2 + K_3 q^\mu q^\nu + K_4 (p^\mu q^\nu + q^\mu p^\nu), \tag{III.124}
\]

with \(K_{(q^2)}\) scalar functions of only \(q^2\) (note that \(p^2 = M^2\) is just a constant, and the scalar \(q^\mu p = -q^2/2\) is proportional to \(q^2\)). Antisymmetric terms like \(p^\mu q^\nu - q^\mu p^\nu\) do not contribute because they do not contribute in the contraction with the symmetric Lepton tensor \(L_{(e)}^\mu\nu.\) Pseudovector/tensor terms cannot come in either because they would violate parity conservation as observed in EM interactions. Finally, the proton tensor can be further simplified using current conservation: \(q_\mu L_{(e)}^\mu\nu = q_\mu L_{(e)}^\nu\nu = 0:\)

\[
K^\mu\nu_{(p)} = K_1 p^\mu p^\nu + K_2 g^\mu\nu q^2 \tag{III.125}
\]
so that only two functions remain. The matrix element for ep scattering may now be calculated

\[
|m_{f\ell}(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{q^4} L_{\mu\nu}^{\mu\nu} K_{\nu\mu\nu} = \frac{e^4}{q^4} \left( 2 p_\mu^\ell p_1^\nu + 2 p_\mu^\ell p_3^\nu + g_\mu^\nu q^2 \right) \left( 4 K_1 p_{2\mu} p_{2\nu} + K_2 g_{\mu\nu} q^2 \right)
\]

\[
= \frac{e^4}{q^4} \left[ 16 \left( (p_1 \cdot p_2) (p_2 \cdot p_3) + 4 m_2^2 q^2 \right) K_1 + 2 \left( m_1^2 + m_3^2 + q^4 \right) K_2 \right]
\]

Neglecting the electron’s mass \((m_1=m_3\approx0)\) and going to the Laboratory System (proton at rest, mass \(M\)), we get:

\[
\frac{d\sigma_{ep\rightarrow ep}}{d\Omega} \bigg|_{\text{LAB}} = \frac{1}{64\pi^2 M^2} \left( \frac{E_3}{E_1} \right)^2 |m_{ep\rightarrow ep}|^2
\]

\[
= \frac{1}{64\pi^2 M^2} \left( \frac{E_3}{E_1} \right)^2 (4\pi\alpha)^2 16 M^2 E_3 E_3 \left[ K_1 \left( 1 + \sin^2 \theta \right) + K_2 \frac{-q^2}{2M^2} \sin^2 \theta \right]_{m=0}
\]

\[
= \frac{\alpha^2}{4E_3^2 \sin^4 \frac{\theta}{2}} \left( \frac{E_3}{E_1} \right) \left[ K_1 \cos^2 \frac{\theta}{2} + K_2 \frac{-q^2}{2M^2} \sin^2 \theta \right]
\]

This can now be compared to point-like spin-\(\frac{1}{2}\) scattering, see \(ep\) elastic scattering in equation (III.111). The fact that \(K_1\) and \(K_2\) are functions of \(q^2\) tells us already that the proton is not point-like. Deep inelastic ep scattering reveals the (charged) sub-structure in the proton (and neutron).
3.8 Summary of Feynman Rules

Below we give a summary of all the Feynman rules we have so far “derived”:

<table>
<thead>
<tr>
<th>Rule 1</th>
<th>Rule 2</th>
<th>Rule 3</th>
</tr>
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<tbody>
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</table>
4 The Weak Interaction

Historically, the theory of weak interactions started with Fermi’s four-fermion point interaction of the form:

\[ -i m_{\beta} = i G_F \left( i \nu_{(\mu)} \gamma^\mu \mu \right) \left( i \nu_{(e)} \gamma^e e \right) \]  

(III.128)

where the Fermi constant \( G_F \) described the strength of the interaction. The Fermi constant has dimension Energy\(^{-2} \), which indicates that this description can only be a low-energy approximation. The lifetime of the muon can be calculated with the above matrix element using the trace techniques developed for QED:

\[ |m_{\beta}|^2 = \frac{1}{2} G_F^2 \text{Tr} \left[ \gamma^\mu p^\mu + m_\mu \right] \text{Tr} \left[ \gamma_\nu \left( p^\nu_4 + m_\nu \right) \gamma_\nu p^\nu_4 \right] = \frac{G_F^2}{2} \left[ 4 p^\nu p^\nu_4 + 4 p^\mu p^\mu_4 - 4 g^\nu_\mu (p^\nu_4 p^\mu_4) \right] \left[ 4 p^\nu_4 p^\nu_4 + 4 p^\mu_4 p^\mu_4 - 4 g^\nu_\mu (p^\nu_4 p^\mu_4) \right] \]  

(III.129)

where the factor \( \frac{1}{2} \) is from the averaging over the muon spin states. The decay width is now calculated as:

\[ d\Gamma_{(1\to2+3+4)} = \frac{1}{\text{Flux}} |m_{\beta}|^2 d\text{Lips} \]  

(III.130)

Next, this is integrated over \( dp^2 \) and \( dp^3 \) using the delta-function, giving:

\[ \frac{d\Gamma}{dE_e} = \frac{G_F^2}{16 \pi} m^2_e E_e \left[ 2 - \frac{4E_e}{m_\mu} + \left( 1 - \frac{4E_e}{3m_\mu} \right) \right] \Rightarrow \Gamma = \frac{m^2_e G_F^2}{384 \pi^2} = (4.38 \mu s)^{-1} \]  

(III.131)

which is twice the measured muon lifetime value of 2.20 \( \mu s \). This long lifetime, similar to many other particles that decay weakly, characterizes the weak interaction and sets it apart from the EM and strong interactions.

It was rapidly recognized by Fermi that the amplitude (III.128) led to problems at much higher energies, and he postulated the existence of the massive intermediate vector boson \( W \) as the mediator of the weak interaction. This explains, at least qualitatively, that the weak cross sections at low energy and the weak decays of particles are suppressed. It also explains that in these situation the \( q^2 \) dependence is absent.

It was later realized, by T.D. Lee and C.N. Yang, that the weak interaction violates parity and the parity violation was subsequently seen to be maximal. It ap-
pears that only left-handed particles participate in the weak interaction. Because neutrinos are almost massless and because they only inter-
act via the weak interaction, they are produced and detected as left-handed.
We can thus apply the whole Dirac formalism developed previously, provided we project out the left-handed particle spinors and right-
handed anti-particle spinors. We do this by operating on the particle spinors with $P_L$ and on the anti-particle spinors with $P_R$!
Furthermore, we replace the electromagnetic coupling $e = (4\pi\alpha)^{-1/2}$ with the weak coupling constant $g/\sqrt{2}$ (the $\sqrt{2}$ is an historical artifact). In the Glashow-Weinberg-Salam model (the electro-weak sector of the Standard Model) $g$ and $e$ are related by the Weinberg angle:

$$g = e \sin \theta_W = 0.481 e$$

(III.132)

As mentioned before, in many low-energy calculations, the explicit $W$-boson propagator reduces to a simple constant $M_W^{-2}$. Thus, the Fermi coupling constant $G_F$, very precisely measured from the muon lifetime, relates to the combination of the weak coupling $g$ and the $M_W$ as:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

(III.133)

In the following sections, we will calculate a number of important cross sections and lifetimes: the muon lifetime, the scattering cross section of muon neutrinos on (atomic) electrons, and the scattering of electron anti-neutrinos on atomic electrons.

### 4.1 Muon Decay

We now possess all ingredients to calculate muon decay. The matrix element, based on Figure 5, is:

$$-iM_{\beta i} = -ig_{\mu i}G_F(\bar{u}_3\gamma^\mu(1-\lambda\gamma_5)u_1)(\bar{u}_4\gamma^\nu(1-\gamma_5)u_2)$$

(III.134)

where $\lambda$ is an a priori unknown parameter that allows us to switch between various assumptions for the weak interaction in the $W_\mu\nu_\mu$ vertex: $\lambda = 0$ for the Fermi model, $\lambda = +1$ for the Glashow-Weinberg-Salam $V-A$ theory, and $\lambda = -1$ for a $V+A$ theory. The matrix element squared, averaged and summed over the initial and final spins respectively, becomes:

$$|\overline{M}_{\beta i}|^2 = \frac{1}{2} \frac{G_F^2}{2} \sum_{s_1,s_2,s_3,s_4} \left[ (\bar{u}_3\gamma^\mu(1-\lambda\gamma_5)u_1)(\bar{u}_4\gamma^\nu(1-\gamma_5)u_2) \right]^2 \left[ (\bar{u}_3\gamma^\mu(1-\lambda\gamma_5)u_1)(\bar{u}_4\gamma^\nu(1-\gamma_5)u_2) \right]^2$$

$$= \frac{G_F^2}{4} \sum_{s_1,s_2,s_3,s_4} \left[ (\bar{u}_3\gamma^\mu(1-\lambda\gamma_5)u_1)(\bar{u}_4\gamma^\nu(1-\gamma_5)u_2) \right]^2 \left[ (\bar{u}_3\gamma^\mu(1-\lambda\gamma_5)u_1)(\bar{u}_4\gamma^\nu(1-\gamma_5)u_2) \right]^2$$

(III.135)
Using the Trace algorithms, see equations (III.119) and/or (III.120), we obtain:

\[
\begin{align*}
\left| m_{\beta} \right|^2 &= \frac{G_F^2}{4} \text{Tr} \left[ \gamma_\mu \not{p}_3 \gamma_\nu \not{p}_1 \left( 1 + \lambda^2 - 2\lambda \gamma_5 \right) \right] \times 2 \text{Tr} \left[ \gamma^\mu \not{p}_4 \gamma^\nu \not{p}_5 \left( 1 - \gamma_5 \right) \right] \\
&= \frac{G_F^2}{4} \left[ (1 + \lambda^2)32 (p_3 p_4 (p_1 p_2) + (p_3' p_2') (p_1 p_4)) + 4\lambda 32 (p_3 p_4 (p_1 p_2) - (p_3 p_2') (p_1 p_4)) \right] \\
&= 16G_F^2 \left[ (1 + \lambda^2) (p_3 p_4 (p_1 p_2) + (p_3' p_2') (p_1 p_4)) + 2\lambda (p_3 p_4 (p_1 p_2) - (p_3 p_2') (p_1 p_4)) \right] \\
&= 16G_F^2 \left[ (1 + \lambda^2) (p_3 p_4 (p_1 p_2) + (1 - \lambda^2) (p_3 p_2') (p_1 p_4)) \right].
\end{align*}
\]

The decay width of the muon is then:

\[
d\Gamma = \frac{1}{2m_1} \frac{(2\pi)^4 dp_3 dp_4 dp_5}{2E_3 2E_2 2E_4} \delta^4 (p_1 - p_2 - p_3 - p_4) \left| m_{\beta} \right|^2,
\]

which we’ll simplify by using the following:

- \[
\int \frac{dp_3 \delta^4 (p_3 + p_2 + p_4 - p_1)}{2E_3} = \frac{1}{2E_2 E_4} \theta(m_1 - E_2 - E_4) \delta(\cos \theta_{24} - ...) 
\]

- \[
\int \frac{dp_2 dp_4}{2E_2 2E_4} = 8\pi^2 \int \frac{E_2^2 E_4^2}{4E_2 E_4} dE_2 dE_4 d\cos \theta_{24} = 2\pi^2 \int E_2 E_4 dE_2 dE_4 d\cos \theta_{24}
\]

- \[
p_1 - p_2 = p_3 + p_4 \Rightarrow (p_1 p_2) (p_3 p_4) = m_1 E_2 \left( \frac{1}{2} m_1^2 - m_1 E_2 \right) \text{ (in muon rest system)}
\]

- \[
p_1 - p_4 = p_2 + p_3 \Rightarrow (p_1 p_4) (p_2 p_3) = m_4 E_4 \left( \frac{1}{2} m_1^2 - m_4 E_4 \right) \text{ (in muon rest system)}
\]

\[
d\Gamma = -\frac{16G_F^2}{64\pi^2 m_1^3} \int dE_2 dE_4 d\cos \theta_{24} \left[ (1 + \lambda)^2 E_2 \left( \frac{1}{2} m_1 - E_2 \right) + (1 - \lambda)^2 E_4 \left( \frac{1}{2} m_1 - E_4 \right) \right] \delta(\cos \theta_{24} - ...)
\]

with limits set by: \(\cos \theta_{24} = +1 \rightarrow E_2 = \frac{1}{2} m_1 - E_4; \cos \theta_{24} = -1 \rightarrow E_2 = \frac{1}{2} m_1:\)

\[
d\Gamma \bigg|_{E_2 = \frac{1}{2} m_1 - E_4} = \frac{G_F^2}{4\pi^2} m_1 \left[ \int_{m_1/2}^{m_1/2} dE_2 \left[ (1 + \lambda)^2 E_2 \left( \frac{1}{2} m_1 - E_2 \right) + (1 - \lambda)^2 E_4 \left( \frac{1}{2} m_1 - E_4 \right) \right] \right]
\]

Change variables: \(x = \frac{1}{2} m_1 - E_2; E_2 = \frac{1}{2} m_1 - x:\)

\[
\frac{d\Gamma}{dE_4} = \frac{G_F^2}{4\pi^2} m_1 \left[ \int_0^{m_1/2} dx \left[ (1 + \lambda)^2 x \left( \frac{1}{2} m_1 - x \right) + (1 - \lambda)^2 E_4 \left( \frac{1}{2} m_1 - E_4 \right) \right] \right] = \frac{G_F^2}{4\pi^2} m_1 \left[ (1 + \lambda)^2 x^2 \left( \frac{1}{2} m_1 - \frac{1}{2} x \right) + (1 - \lambda)^2 E_4 x \left( \frac{1}{2} m_1 - E_4 \right) \right]^{E_4}_{0}
\]

\[
= \frac{G_F^2}{4\pi^2} m_1 E_4^2 \left[ (1 + \lambda)^2 \frac{1}{2} \left( \frac{3}{2} m_1 - E_4 \right) + (1 - \lambda)^2 \left( \frac{1}{2} m_1 - E_4 \right) \right] = \frac{G_F^2}{8\pi^2} m_1 E_4^2 \left[ (1 + \lambda)^2 \frac{1}{2} \left( 1 - \frac{4E_4}{3m_1} \right) + (1 - \lambda)^2 \left( 1 - \frac{2E_4}{m_1} \right) \right]
\]

(III.136)
This is often expressed in terms of the “Michel parameter” \( \rho = (3/8) (1 + \lambda)^2 / (1 + \lambda^2) \) and \( x = E / E_{\text{max}} = 2E / m_\mu \):

\[
\frac{d\Gamma}{dx} = \frac{G_F^2 m_\mu^5}{96\pi^3} \left( 1 + \lambda^2 \right)^2 \left( 1 - x \right) \left[ 3(1-x) + \frac{2}{3} \rho \left( 4x - 3 \right) \right]
\]

The total decay rate becomes:

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

\[
\frac{d\Gamma}{dx} = \frac{G_F^2 m_\mu^5}{96\pi^3} \left( 1 + \lambda^2 \right)^2 \left( 1 - x \right) \left[ 3(1-x) + \frac{2}{3} \rho \left( 4x - 3 \right) \right]
\]

The total decay rate becomes:

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

\[
\text{(III.137)}
\]

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

The total decay rate becomes:

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

The total decay rate becomes:

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

The total decay rate becomes:

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

The total decay rate becomes:

\[
\Gamma (\mu \to \nu_\mu + e^- + \nu_e) = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \int_0^{m_{\mu}/2} E' dE' \left[ (1+\lambda)^2 \left( 1 - \frac{4E_4}{3m} \right) + (1-\lambda)^2 \left( 1 - \frac{2E_4}{m} \right) \right] = \frac{G_F^2 m_\mu^2}{8\pi^2} \int_0^{m_{\mu}/2} E^2 dE \left[ 1 + \frac{4E_4}{3m} \right] \]

Indeed, for the \( V - A \) theory \( (\lambda = 1) \) we find excellent agreement with the experimental value of 2.1970 \( \mu s \).

### 4.2 Weak Decay of the Charged Pion

The charged members of the \( J^P = 0^- \) pseudoscalar nonet decay weakly. We may calculate their expected decay rates in a way similar to muon decay. The Feynman diagram is shown in Figure 6. The matrix element involves the \( \nu_e \to e^- \) left handed current and the pionic quark current \( J^\mu (q) \):

\[
-iM_{fi} = -ig_\mu \frac{G_F}{\sqrt{2}} u_e(p_1) J^\mu (q) u_{\nu_e}(-p_2) J^\nu (q)
\]

Here, the quark current is “hidden” inside the pion. The only variable that is available to make a Lorentz four-current is the vector \( q^\mu \). For a spin-zero pion, the current must thus be of the form \( J^\mu (q) = q^\mu f(q^2) \) which, for the because \( q^2 = m_\pi^2 \), simplifies to \( J^\mu (q) = q^\mu f(q^2) = q^\mu f_\pi \). Combining, we can calculate the pion decay matrix element:

\[
M_{fi} = f_\pi \frac{G_F}{\sqrt{2}} u_e(p_1) J^\mu (q) u_{\nu_e}(-p_2) = f_\pi \frac{G_F}{\sqrt{2}} (p_1 + p_2) J^\mu (q) (\gamma^\mu - q^\mu) u_{\nu_e}(-p_2) = f_\pi \frac{G_F}{\sqrt{2}} (p_1 + p_2) J^\mu (q) (\gamma^\mu - q^\mu) u_{\nu_e}(-p_2) = f_\pi \frac{G_F}{\sqrt{2}} (p_1 + p_2) J^\mu (q) (\gamma^\mu - q^\mu) u_{\nu_e}(-p_2)
\]

The squared and spin-summed matrix element becomes:

\[
W^* e^- (p_1) \rightarrow \nu_e (p_2) \rightarrow e^- (p_1) \rightarrow \nu_e (p_2)
\]

Figure 6. Feynman diagram for the charged pion decay via the weak interaction.
\[
\left| m_{\mu} \right|^2 = \frac{m_e^2 f_\pi^2 G_F^2}{2} \sum_{q_1, q_2} \left( \bar{u}_e(p_1)(1 - \gamma^5)u_{\nu_e}(-p_2) \right) \left( \bar{u}_e(p_1)(1 - \gamma^5)u_{\nu_e}(-p_2) \right)^* \\
= m_e^2 f_\pi^2 \frac{G_F^2}{2} \text{Tr} \left[ \left( \rho_1'(1 + \gamma^5) \right) \rho_2'(1 + \gamma^5) \right] = 4m_e^2 f_\pi^2 G_F^2 (p_1 \cdot p_2)
\]

In the pion CMS the product \(p_1 p_2\) reduces to \(\frac{1}{2}(s - m_1^2 - m_2^2) = \frac{1}{2}(m_\pi^2 - m_e^2)\), a simple constant. Finally, the decay width is calculated using (II.46):

\[
\Gamma(\pi^- \rightarrow e^- + \bar{\nu}_e) = \frac{1}{32\pi^2 m_\pi^2} |p^*| \left\vert \int d\Omega^* \left| m_{\mu \rightarrow e\nu_e + \bar{\nu}_e} \right|^2 \right\vert = \frac{1}{32\pi^2 m_\pi^2} \frac{m_e^2 - m_\pi^2}{2m_\pi} 4\pi \left[ 2m_e^2 f_\pi^2 G_F^2 \left( m_\pi^2 - m_e^2 \right) \right]
\]

\[
= \frac{f_\pi^2 G_F^2}{8\pi} \frac{m_\pi^2}{m_\pi} \left( m_\pi^2 - m_e^2 \right)^2 = \frac{f_\pi^2 G_F^2}{8\pi} \frac{m_\pi m_e^2}{m_\pi} \left( 1 - \frac{m_e^2}{m_\pi^2} \right)^2
\]

(III.138)

### 4.3 Inverse Muon Decay: Muon Neutrino Scattering Off Electrons

Neutrino beams can be produced at high-energy, high-intensity proton accelerators. Accelerated protons are extracted and directed onto a metal target (cooling!). The great majority of particles produced in the target are pions, and the charged pions are allowed to decay in a beam (tunnel) section behind the target up to several hundred meters in length. The charged pions decay in-flight into muon neutrinos (with a 100% branching fraction); a \(\pi^+\) into a muon neutrino, and a \(\pi^-\) into a muon anti-neutrino. The neutrinos are Lorentz boosted in the direction of the pions. The accompanying muons that are also produced are absorbed in a long beam stop (a long dirt-filled section of the beam line) so that only the neutrinos survive. One may even select the wanted neutrino type by selectively bending away either the positive or the negative pions...

A very large detector, e.g., a water Cerenkov detector, or a detector with liquid scintillator planes alternating with tracking planes and magnetized iron absorbers, follows the muon filter.

The scattering of muon neutrinos, the left diagram in Figure 7, is directly related to muon decay, the right diagram in Figure, via a topological rearrangement of fourvectors. Interactions that have \(W\)-boson exchange are called “Charged Current” events, as opposed to events that result from the exchange of a \(Z\)-boson, which are called “Neutral Current” events; these latter events were feverishly searched for when the Glashow-Weinberg-Salam model predicted the \(Z\) as well as the \(W\).

Borrowing directly from the muon decay result, we have:

![Figure 7. Feynman diagram for muon neutrino - electron scattering (left), and its relationship with the muon decay diagram (right).](image-url)
\[
|m_{\ell}|^2 \left( \frac{g^2}{8M_W^2} \right)_{\nu_\mu \to \nu_e; 1 \to 2 \to 3 \to 4} = 128(p_1 p_2)(p_3 p_4) \Rightarrow
|m_{\ell}|^2 \left( \frac{g^2}{8M_W^2} \right)_{\nu_\tau \to \nu_e; 1 \to 2 \to 3 \to 4} = 128((-p_3 p_4)(-p_1 p_2) = \left( \frac{g^2}{8M_W^2} \right)_{2} 128(p_3 p_4)(p_1 p_2)
\]
(III.139)

This is independent of any angles: the dot products \(p_1 p_2\) and \(p_3 p_4\) are only proportional to the total CMS energy squared. Therefore, the angular distribution of the muon and electron neutrino in the final state are isotropic in the CMS:
\[
\frac{d\sigma(v_\mu e^{-}\to \nu_e \mu^+)}{d\Omega} = \frac{1}{64\pi^2 s} G^2 \frac{s}{4\pi^2}; \quad \sigma(v_\mu e^{-}\to \nu_e \mu^+) = \frac{G^2 2m_{\ell} E_{\ell}^{\text{lab}}}{\pi} = s \times 1.68 \times 10^{-11} \text{ mb GeV}^{-2} = E_{\nu}^{\text{lab}} \times 1.72 \times 10^{-14} \text{ mb/GeV}
\]
(III.140)
i.e. very very small! For example, a 10 GeV beam of \(10^{10}\) neutrinos per second, entering a 100 ton H\(_2\)O target will produce only about 21 interactions per hour. Note that the muon mass cannot be ignored at current beam energies:
\[
m_{\mu}^2 = 2m_{\ell} E_{\ell}^{\text{lab}}
\]
is not large compared to \(m_{\mu}^2\)!
Note, that this ever-increasing cross section is unphysical in the limit of ultra-high energy, it violates Unitarity. Thus, at some energy, a new process must come in, another Feynman diagram, that moderates this behavior of the cross section.

### 4.4 Electron Anti-Neutrino scattering off Atomic Electrons, the Z-Boson

Electron (anti-)neutrinos are copiously produced in natural and man-made nuclear reactors. When scattering off atomic electrons, two diagrams play a role, a charged current diagram and a neutral current diagram, see Figure 8.

#### 4.4.1 The Charged Current Diagram

The calculation of the first diagram simply follows from the muon decay diagram of Figure 7 with the substitutions: \(p_1 \to -p_3\), \(p_3 \to -p_1\) and/or the interchange of the Lorentz invariants \(s\) and \(t\). In the CMS of the \(v_e e^-\) system we find, see (III.140):
\[
\frac{d\sigma(v_e e^{-}\to \nu_e e^-)}{d\Omega} = \frac{1}{64\pi^2 s} |m_{\ell}|^2 = \left( \frac{1}{64\pi^2 s} \right) 16G_F^2 t^2
\]
(III.141)

In terms of CMS quantities and ignoring the (small) masses:
\[
s = 2p_1 \cdot p_2 = 4E^2
\]
\[
t = -2p_1 \cdot p_3 = -2E^2(1 - \cos \theta_{\nu_e e^-})
\]
\[
u = -2p_1 \cdot p_4 = -2E^2(1 - \cos \theta_{\nu_e e^-})
\]
we obtain:

\[
\begin{align*}
\begin{cases}
s = 2p_1 \cdot p_2 = 4E^2 \\
t = -2p_1 \cdot p_3 = -2E^2(1 - \cos \theta_{\nu_e e^-}) \\
u = -2p_1 \cdot p_4 = -2E^2(1 - \cos \theta_{\nu_e e^-})
\end{cases}
\end{align*}
\]
(III.142)
\[
\frac{d\sigma(v_e^- \to e^- v_e)}{d\Omega} = \frac{G_F^2 s}{16\pi^2} \left(1 - \cos \theta_{v_e^-}\right)^2; \quad \sigma(v_e^- \to e^- v_e) = \frac{G_F^2 s}{3\pi} = \frac{1}{3} \sigma(v_\mu^- \to \mu^- v_\mu)
\] (III.143)

The angular cross section peaks for \(\theta(v_e^-) = \pi\), i.e. the outgoing electron emitted opposite in direction to the incoming neutrino beam. This maximal parity violation effect can also be argued based on lefthanded neutrinos and simple spin arguments.

### 4.4.2 The Neutral Current Diagram

We note that the neutral current diagram in Figure 8 is very similar to \(v_\mu e^- \to v_\mu e^-\) or \(\bar{v}_\mu e^- \to \bar{v}_\mu e^-,\) which both only proceed via Z-exchange in the “t-channel”, see Figure 9. Immediately we encounter a problem: the \(v_\mu \to v_\mu\) current will have the \(V-A\) structure because interacting neutrinos are exclusively lefthanded; but what about the \(e^- \to e^-\) current? Also, what is the coupling strength to the \(Z\)-boson? As a first guess we may assume that the \(Z\) couples exactly like the \(W\) to lefthanded leptons; then \(\sigma(v_\mu e^- \to v_\mu e^-) = \sigma(v_\mu e^- \to \mu^- v_e)\). Experimentally however, \(\sigma(v_\mu e^- \to v_\mu e^-) = E_{\text{lab}} (1.6 \pm 0.4) \times 10^{-15} \text{mb/GeV},\) i.e. about ten times smaller.

Therefore, we cannot assume anything and must parametrize our ignorance:

\[
-i m_\mu = -\frac{i \rho G_F}{\sqrt{2}} \left(\hat{S}_3 \gamma^\mu (1 - \gamma_5) v_1 \right) \left(i e_4 \gamma_\mu (c_\tau - c_\alpha \gamma_5) e_2\right),
\] (III.144)

with \(\rho = 1\) if the \(Z\) and the \(W\) couple with the same strength to leptons and have the same mass, and with \(c_\tau = c_\alpha = 1\) for pure \(V-A\) coupling at the electron vertex. The neutrino-\(Z\) tensor is as before for the \(W\), see (III.135):

\[
L_{\nu e}^\nu = \frac{1}{2} \text{Tr} \left[ \gamma_\mu q_3 \gamma_\nu p_4 (1-\gamma_5) q_1 \right] = \frac{1}{2} \text{Tr} \left[ \gamma_\mu q_3 \gamma_\nu p_1 q_4 \right] - \frac{1}{2} \text{Tr} \left[ \gamma_\mu q_3 \gamma_\nu p_4 q_1 \right] = 4 \left[ p_3 p_1 + p_3 p_2 - g_\mu p_1 \cdot p_3 \right] - 4 \left[ -i e_\mu e_\nu p_5^\mu p_5^\nu \right]
\] (III.145)

whereas the electron-\(Z\) tensor becomes:

\[
L_{e e}^\nu = \frac{1}{2} \text{Tr} \left[ \gamma^\mu (c_\tau - c_\alpha \gamma_5) q_4 \gamma^\nu (c_\tau - c_\alpha \gamma_5) q_2 \right] = (c_\tau - c_\alpha)^2 \frac{1}{2} \text{Tr} \left[ \gamma^\mu \gamma_4 \gamma^\nu \gamma_2 \right] + 2 c_\tau c_\alpha \frac{1}{2} \text{Tr} \left[ \gamma_\mu \gamma_4 \gamma_\nu \gamma_2 \right] = (c_\tau - c_\alpha)^2 \left[ 2 p_4^\mu p_2^\nu + 2 p_5^\nu p_4^\mu - 2 g_\mu p_2 \cdot p_4 \right] + 2 c_\tau c_\alpha \left[ -2 i e_\mu e_\nu p_4 p_2 p_2 \right]
\] (III.146)

and the averaged matrix element squared is:

\[
\left| m_\nu \right|^2 = \left( \frac{\rho G_F}{\sqrt{2}} \right)^2 \left( 32 (c_\tau + c_\alpha)^2 \left[ (p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3) \right] + 32 (2 c_\tau c_\alpha) \left[ (p_1 p_2)(p_3 p_4) - (p_1 p_4)(p_2 p_3) \right] \right)
\] (III.147)

The CMS angular distribution of the outgoing electron in the neutral current interaction is then (see (III.142)):
\[
\frac{d\sigma(\nu_\mu e^- \rightarrow \nu_\mu e^-)}{d\Omega_{14}} = \frac{1}{64\pi^2 s} \left| m_\mu \right|^2 = \frac{1}{64\pi^2 s} 16 (\rho G_F)^2 \left(\frac{s}{2}\right)^2 \left[ (c_\rho^2 + c_\alpha^2) + (c_\rho^2 - c_\alpha^2) \frac{1}{4} (1 - \cos \theta_{14})^2 \right] \\
= \frac{(\rho G_F)^2 s}{16\pi^2} \left[ (c_\rho^2 + c_\alpha^2) + \frac{1}{4} (c_\rho^2 - c_\alpha^2) (1 - \cos \theta_{14})^2 \right] \\
= \frac{(\rho G_F)^2 s}{4\pi} \left[ (c_\rho^2 + c_\alpha^2) + \frac{1}{3} (c_\rho^2 - c_\alpha^2) \right] \\
\tag{III.148}
\]

4.4.3 Electron Neutrino Scattering off Electrons

In the calculation of the full process for $\nu_e e^- \rightarrow \nu_e e^-$ we must combine amplitudes before squaring and thus the relative sign of the two contributing diagrams in Figure 8 is crucial. It can be shown that the relative sign of the amplitudes is negative, e.g. by realizing that the neutral current diagram has an interchange of the two final state fermions and thereby acquires an extra $-$ sign.

The various neutrino experiments measure $c_\nu$ and $c_A$ in various combination and give excellent agreement with the Standard Model expectations $c_\nu = I_3 - 2Q \sin^2 \theta_W$ and $c_A = I_3$, where $I_3$ is the $z$-component of weak isospin, $\theta_W$ the Weinberg angle, and $Q$ the lepton charge.