Honors Classical Physics I

PHY141
Lecture 22
Dynamics of Rotations (End)
Statics: equilibrium
Summary Extended objects

- Center-of-Mass: \( \mathbf{r}_{CM} \equiv \sum_i m_i \mathbf{r}_i / M = \int \mathbf{r} dm / M \Rightarrow \mathbf{v}_{CM} = \sum_i m_i \mathbf{v}_i / M, \quad \mathbf{a}_{CM} = \ldots \)

<table>
<thead>
<tr>
<th>Linear</th>
<th>using ...</th>
<th>Rotational</th>
<th>using ...</th>
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</table>
| \( s(t), \quad \mathbf{v}(t) \equiv ds/dt, \)  
\( \mathbf{a}(t) \equiv d\mathbf{v}/dt \ldots \) | \( \theta = s(t)/R \) | \( \theta(t), \)  
\( \mathbf{\omega}(t) \equiv d\theta/dt, \)  
\( \mathbf{\alpha}(t) \equiv d\mathbf{\omega}/dt \ldots \) |  |
| \( \mathbf{F}_{net} = m\mathbf{a} = d\mathbf{p}/dt \) | \( \mathbf{p} \equiv m\mathbf{v} \) | \( \mathbf{\tau}_{net} = I\mathbf{\alpha} = d\mathbf{L}/dt \) | \( \mathbf{\tau}_{R,F} \equiv \mathbf{R} \times \mathbf{F}, \)  
\( I \equiv \int r^2 dm, \)  
\( \mathbf{L} \equiv \mathbf{R} \times \mathbf{p} = I\mathbf{\omega} \) |
| \( \Rightarrow \mathbf{J} \equiv \int_{\Delta t} \mathbf{F}_{net} dt = \Delta \mathbf{p} \) |  |  |  |
| \( W_{net} \equiv \int_{\text{path}} \mathbf{F}_{net} \cdot ds = \Delta \mathbf{K} \) | \( \mathbf{F}_{net} = m\mathbf{a}, \)  
\( K \equiv \frac{1}{2}mv^2 \) | \( W_{net} = \int_{\Delta \theta} \mathbf{\tau}_{net} \cdot d\theta = \Delta \mathbf{K} \) | \( K \equiv \frac{1}{2}I\omega^2 \) |

\[ \mathbf{\tau}_{R,F} = \mathbf{R} \times \mathbf{F} \equiv \begin{vmatrix} i & j & k \\ R_y F_z - R_z F_y & R_z F_x - R_x F_z & R_x F_y - R_y F_x \end{vmatrix} = \begin{vmatrix} i & j & k \\ R_x & R_y & R_z \\ F_x & F_y & F_z \end{vmatrix} \]

Note the circular sequence x, y, z, x, ...
I (100 kg) walk past a lamp post; I pass the lamp post at 2 m, with a speed of 4 m/s. My angular momentum with respect to the lamp post is ...
Bicycle Wheel

Consider a bicycle wheel with moment of inertia $I_{cm} = MR^2$ (all mass in the rim) for rotation axis $\omega$ through cm, ⊥ wheel, which is also a symmetry axis

1. When the wheel isn’t spinning ($L=\omega=0$):

   - the weight exerts net torque:
     \[ \tau_{Net} = \mathbf{r} \times \mathbf{w} = dL/dt = I_y \alpha, \]
     where $I_y$ is the moment of inertia for the rotation axis through $O$, perpendicular to paper.

For a FLAT object in the z-y plane:

- Thus: $I_x = MR^2 = I_y + I_z = 2I_y$, and $I_y = \frac{1}{2}I_x + Mr^2 = \frac{1}{2}MR^2 + Mr^2$

- And the wheel will tip down: $\tau_{net} = \mathbf{r} \times \mathbf{w} = dL/dt = I_y \alpha(\mathbf{j})$
2. When the wheel \textit{is} spinning (\(L = I\omega\)):

\begin{itemize}
  \item the weight still exerts a net torque:
    \[
    \tau_{\text{net}} = r \times w = dL/dt = I_y \alpha(j),
    \]
  \item where \(dL/dt\), as before, is directed along the positive \(y\)-axis, i.e. through \(O\), into the paper.
  \item However, now \(L = L_x \neq 0\), and \(dL = dL_y\) adds to it in time \(dt\), i.e. \(L\) will rotate around \(O\), changing direction, but constant in magnitude!
  \item Thus: instead of tipping down, \textbf{the spinning wheel executes horizontal circles around \(z\) (\(\equiv\) \text{PRECESSION}) with angular velocity:}
    \[
    \omega_z = \frac{d\phi}{dt} = \frac{dL/L}{dt} = \frac{dL/dt}{L} = \frac{\tau_y}{I_x \omega_x},
    \]
  \item The faster \(\omega_x\), the slower precession...
  \item \textbf{We assumed} \(\omega_z \ll \omega_x\), so that \(L = L_z + L_x \approx L_x\)
\end{itemize}
Summary

- **Center-of-Mass:** \( \mathbf{r}_{\text{cm}} \equiv \Sigma m_i \mathbf{r}_i / M \)
- **Moment of Inertia I:** \( I \equiv \int r^2 dm = \Sigma m_i r_i^2 \)
  - Parallel Axis Theorem: \( I_d = I_{/\text{cm}} + Md^2 \)
  - Perpendicular Axis Theorem: for a FLAT object in the \( x-y \) plane: \( I_z = I_x + I_y \)
- **Kinetic Energy:**
  - “fixed axis” (in inertial system): \( K = \frac{1}{2} I \omega^2 \)
  - moving with cm axis' direction fixed: \( K = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2 \)
- **Gravitational Potential Energy of Extended body:** \( U_G = Mgh_{\text{cm}} \)
- **Torque vector:**
  - Net Torque: \( \tau_{\text{Net}} = \mathbf{R} \times \mathbf{F} = I \alpha \)
  - Cross Product:
    \[
    \tau = \mathbf{R} \times \mathbf{F} \equiv \mathbf{i} \left( R_y F_z - R_z F_y \right) + \mathbf{j} \left( R_z F_x - R_x F_z \right) + \mathbf{k} \left( R_x F_y - R_y F_x \right) = \begin{vmatrix} i & j & k \\ R_x & R_y & R_z \\ F_x & F_y & F_z \end{vmatrix}
    \]
    Note the circular sequence \( x, y, z, x, ... \)

- **Angular Momentum:** \( \mathbf{L} \equiv \mathbf{R} \times \mathbf{p} \) and \( d\mathbf{L}/dt = \tau_{\text{Net}} \)
  - If \( \tau_{\text{Net}} = 0 \) then \( \mathbf{L} \) is conserved!
- **Precession...** \( \omega_z = \frac{d\varphi}{dt} = \frac{d\mathbf{L}/L}{dt} = \frac{d\mathbf{L}/dt}{I_x \omega_x} = \frac{\tau_y}{I_x \omega_x} \)
Equilibrium

• Many types of constructions (i.e. extended and rigid objects) are meant to be stable and unmoving: buildings, bridges, dams, signposts, etc.

• In such cases both the $a_{cm}$ and the angular acceleration $\alpha$ (around ANY axis!) must be zero:

  - EQUILIBRIUM:

    $$ F_{\text{Net}} = \sum_i F_i = 0 \quad \text{and} \quad \tau_{\text{Net}} = \sum_i \tau_i = 0 $$

  - In two dimensions this gives us (typically) a total of three equations ($x$ and $y$ from the first, and $z$ from the second equation)!

    • Thus, we can solve for 3 quantities (force components, etc.)
Wikipedia: On the evening of July 17, 1981, approximately 1,600 people gathered in the atrium to participate in and watch a dance competition. Many people stood on the two connected walkways. At 7:05 p.m. the second-level walkway held approximately 40 people with more on the third and an additional 16 to 20 on the fourth level who watched the activities of the crowd in the lobby below. The fourth floor bridge was suspended directly over the second floor bridge, with the third floor walkway offset several meters from the others. Construction difficulties resulted in a subtle but flawed design change that doubled the load on the connection between the fourth floor walkway support beams and the tie rods carrying the weight of both walkways. This new design was barely adequate to support the dead load weight of the structure itself, much less the added weight of the spectators. The connection failed, and the fourth-floor walkway collapsed onto the second-floor walkway. Both walkways then fell to the lobby floor below, resulting in 111 deaths at the scene and 219 injuries.

- What was the fatal flaw?
Example

A shop sign hangs as indicated in the picture. The mass of the chain can be ignored.

- Calculate $T$ and $F$

\[ \tau_\text{O}: \quad \left( \frac{L}{2} \right) mg + \left( L - \frac{l}{2} \right) (Mg - T \sin 30^\circ) = 0 \]

\[ \Rightarrow \quad T \sin 30^\circ = \frac{L}{2L-l} mg + Mg = \left( \frac{mL}{2L-l} + M \right) g \]

$y$-dir: \[ F_y + T \sin 30^\circ - (m + M) g = 0 \]

\[ \Rightarrow \quad F_y = (m + M) g - T \sin 30^\circ = \frac{L-l}{2L-l} mg \]

$x$-dir: \[ F_x - T \cos 30^\circ = 0 \]

\[ \Rightarrow \quad F_x = T \cos 30^\circ = \frac{T \sin 30^\circ}{\tan 30^\circ} = \ldots \]
Example

A ladder, mass \( m = 30 \text{ kg} \) and length \( L = 6 \text{ m} \), makes an angle 20° with the vertical. The coefficient of static friction between floor and ladder is 0.25, and the friction between ladder and wall is zero ...

- How far can a person of mass \( M = 80 \text{ kg} \) climb up before the ladder starts slipping?

- Equations for equilibrium (unknowns in red):

  \[ \tau_A: \quad xMg \sin 160° + \left( \frac{L}{2} \right) mg \sin 160° - LN_w \sin 70° = 0 \]

  \[ = xMg \sin 20° + \left( \frac{L}{2} \right) mg \sin 20° - LN_w \cos 20° = 0 \]

  \( x \)-dir: \( \mu_s N_F - N_W = 0; \quad y \)-dir: \( N_F - mg - Mg = 0 \)

- From 3rd: \( N_F = (m+M)g \)

- into 2nd: \( N_W = F_s \leq \mu_s N_F = \mu_s (m+M)g \)

- into 1st:

  \[ \frac{x}{L} \leq \left[ \mu_s \frac{M + m}{M \tan 20°} - \frac{m}{2M} \right] = [0.944 - 0.188] = 0.757 \]

  \[ \Rightarrow x \leq 4.54 \text{ m} \]

- and once it starts sliding, kinetic friction takes over!
**Free Rotations**

Free Rotation is rotation where the rotation axis is not fixed and where NO external forces are directly acting on the axis

- Free rotation must be around the Center-of-Mass (CM) of the system ...
  - otherwise an external force would be needed (from CM to the rotation axis) to keep the CM going through a circle!
  - that would be a centripetal force...
- similarly, it is easy to see that it must be around a symmetry axis:
  - an external torque would be necessary if not...
I throw a wrench towards my toolbox; the *location* of the axis of rotation of the wrench when it flies through the air ...

A. depends on how I throw it; e.g. swinging it around its *back end* while releasing it will make it rotate around its *back end* in the air ...

B. I can make it rotate in the air around any point of my choice...

C. is *always* through the CM of the wrench ...
A uniform stick \((M, L)\) is laying on ice and a blob of putty \((m, v_0)\) hits it perpendicularly, a distance \(D\) away from the stick’s center.

- Calculate the \(v_{CM}\) and angular velocity \(\omega\) of the stick+putty after the collision.

- Free rotation: system will rotate around its (new) \(CM'\) after the collision!

\(- x_{CM}:
\begin{align*}
x_{CM} &= \frac{M \times 0 + m \times D}{M + m} = \frac{m}{M + m} D
\end{align*}

- Momentum conservation (axis is NOT fixed):
\[
p_i = p_f \implies mv_0 = (M + m)v_{CM} \implies v_{CM} = \frac{m}{M + m} v_0
\]

- Angular Momentum conservation:
\[
L_i = L_f \implies (D-x_{CM})mv_0 = \left[\frac{1}{12}ML^2 + Mx_{CM}^2 + m(D-x_{CM})^2\right]\omega_{CM} \implies \omega_f = \ldots
\]

- Check: when \(D=0\) \(x_{CM}=0\) and \(\omega_{CM}=0\)
Moving beam and man on ice

A uniform beam \((M, L, v_0)\) is sliding on ice towards a man \((m)\) whom it hits perpendicularly; the man grabs the beam at the end

- Calculate the \(v_{CM}\) and angular velocity \(\omega_{CM}\) of the beam+man after the collision
- Free rotation: system will rotate around its (new) \(CM'\) after the collision!
- \(x_{CM}\):
  \[
x_{CM} = \frac{M \times 0 + m \times L/2}{M + m} = \frac{m}{M + m} \frac{L}{2}
\]
- Momentum conservation (axis is NOT fixed):
  \[
p_i = p_f \implies Mv_0 = (M + m)v_{CM} \implies v_{CM} = \frac{M}{M + m} v_0
\]
- Angular Momentum conservation:
  \[
  L_i = L_f \implies \left(\frac{L}{2} - x_{CM}\right)mv_0 = \left[\frac{1}{12}ML^2 + Mx_{CM}^2 \right. + \left. m\left(\frac{L}{2} - x_{CM}\right)^2\right] \omega_{CM} \implies \omega_f = ...
  \]
Baseball bat’s Sweet Spot …

“Sweet Spot” is the spot \((D)\) where a perpendicular kick causes pure rotation around the handle \((A)\)

- Calculate sweet spot for a bat...
- Sweet Spot: after kick, the \(v_{CM}\) and \(\omega_{CM}\) together produce rotation around the stationary point \(A\) ...
- Kick (Impulse): \(\int Fdt = \Delta p = m v_{CM} \); \[\int Fdt = \Delta p = m v_{CM}\]

\[(D+d-x_{CM}) \times \int Fdt = (D+d-x_{CM}) \int Fdt = \Delta L = I_{CM} \omega_{CM} = I_{CM} \frac{v_{CM}}{x_{CM}-d}\]

- i.e. need to calculate \(I_{CM}\) and \(x_{CM}\) to find \(D\):

\[
m = \int dm = \int_0^L \lambda x dx = \frac{1}{2} \lambda L^2 \Rightarrow \lambda = \frac{2m}{L^2}\]

\[
x_{CM} = \frac{1}{m} \int_0^L \lambda x^2 dx = \frac{1}{3} \lambda L^3 \Rightarrow \frac{1}{3} \lambda L^3 = \frac{2}{3} L \]

\[
I_{O} = \int x^2 dm = \int_0^L \lambda x^3 dx = \frac{1}{4} \lambda L^4 \Rightarrow \frac{1}{2} mL^2 = I_{CM} + mx_{CM}^2 \Rightarrow I_{CM} = \frac{1}{18} mL^2
\]

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"Sweet Spot" is the spot \((D)\) where a perpendicular kick causes pure rotation around the handle \((A)\)

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- Sweet Spot: after kick, the \(v_{CM}\) and \(\omega_{CM}\) together produce rotation around the stationary point \(A\) ...
- Kick (Impulse): \(\int F dt = \Delta p = m v_{CM}\)

\[
\left| (D + d - x_{CM}) \times \int F \, dt \right| = (D + d - x_{CM}) \int F \, dt = \Delta L = I_{CM} \omega_{CM} = I_{CM} \frac{v_{CM}}{x_{CM} - d}
\]

- i.e. need to calculate \(I_{CM}\) and \(x_{CM}\) to find \(D\):

\[
m = \int dm = \int \left( a + bx^2 \right) dx = aL + \frac{1}{3} bL^3
\]

\[
\frac{dm}{dx} = a + bx^2 \quad \Rightarrow \quad dm = (a + bx^2) \, dx
\]

\[
x_{CM} = \frac{\int x \, dm}{\int dm} = \frac{1}{m} \int_a^L \left( a + bx^2 \right) x \, dx = \frac{1}{m} \left( \frac{1}{2} aL^2 + \frac{1}{4} bL^4 \right)
\]

\[
I_O = \int x^2 \, dm = \int_a^L (a + bx^2) x^2 \, dx = \frac{1}{3} aL^3 + \frac{1}{5} bL^5 = I_{CM} + mx_{CM}^2 \quad \Rightarrow \quad I_{CM} = ...
\]
A car \((M=1000 \text{ kg})\) uses a flywheel (disk, \(m, R\)) for energy storage. The range must be \(L=350 \text{ km}\) between recharges, and it makes \(N=20\) accelerations from 0 to \(v=20 \text{ m/s}\). Assume the average drag and friction force is \(F_d=450 \text{ N}\). Assume that energy lost in climbing hills is recovered on the way down...

- calculate the stored energy in the flywheel at the beginning ....

\[
W_{NC} = -F_d L + N \left( -\frac{1}{2} M v^2 \right) = \Delta E = E_f - E_i = 0 - E_i \quad \Rightarrow \quad E_i = F_d L + \frac{N}{2} M v^2
\]

\(E_i \approx 1.6 \times 10^8 \text{ J}\)

- calculate the initial angular speed of the wheel \((m=200 \text{ kg}, R=1.5 \text{ m})\)

\[
E_i = \frac{1}{2} I_{CM} \omega_i^2 = \frac{1}{4} m R^2 \omega_i^2 \quad \Rightarrow \quad \omega_i = \frac{4}{R} \sqrt{\frac{E_i}{m}} \approx 3 \times 10^4 \text{ rad/s}
\]

- how long does it take to charge using a \(P=100 \text{ hp}=75 \text{ kW}\) motor?

\[
P = \frac{dW}{dt} \quad \Rightarrow \quad \Delta t = \frac{W}{P_{\text{avg}}} = \frac{E_i}{P_{\text{avg}}} \approx 1.6 \times 10^8 \text{ s(!)}
\]