Honors Classical Physics I

PHY141
Lecture 21
Rotational Dynamics

Please set your Clicker Channel to 21
Examples

Consider the infamous Yo-Yo (hoop with string wound around it...):

Q: Calculate the tension $T$ in the string:
   - Net force: $T - Mg = Ma_{cm}$
   - Net torque: $\tau = RT = I_{\text{hoop, cm}}a_{cm} = (MR^2)a_{cm}/R$

- This tells us about the MAGNITUDE of $T$,
  ➢ $T$ MUST be opposite from $a_{cm}$ (a string cannot “push”!)
    • Combining $T - Mg = Ma_{cm}$ and $T = -Ma_{cm}$ gives:
      $-Mg = 2Ma_{cm}$ ➢ $a_{cm} = -g/2$ and $T = Mg/2$

Q: Can we do this another way?
   - We can take the (instantaneous) rotation axis to be at $A$:
   - Then:
     $$\tau_{net} = RMg = I_A \alpha = \left( I_{CM} + MR^2 \right) \frac{a_{CM}}{R} = 2MR^2 \frac{a_{CM}}{R} \Rightarrow a_{CM} = \frac{g}{2}$$
a thin uniform vertical stick of length $L$ starts to topple over; its bottom stays in place during the drop. In order to determine the angular velocity $\omega$ of the stick as function of the angle $\theta$ with the horizontal, we should use ...

A. determine the torque as function of $\theta$ and then get the angular acceleration $\omega$ by integration ...

B. use energy conservation ...

C. use $F=ma$, get the speed by integration, and then use $\omega=v/R$ ...

![Diagram of a thin uniform vertical stick with a rotation point labeled at the bottom of the stick.](image)
Toppling Stick …

Q: Find an expression for the angular speed of the stick as function of the angle $\theta$

- Using torque:
  \[ \tau = \frac{L}{2} \cos \theta \ mg = I \alpha = I \frac{d^2 \theta}{dt^2} \quad \Rightarrow \quad \theta = ?? \]
  • not a trivial equation to solve!

- Using energy conservation:
  \[ E_i = E_\theta \Rightarrow \ mg \frac{L}{2} = mg \frac{L}{2} \sin \theta + \frac{1}{2} I \omega^2 \quad \Rightarrow \quad \omega^2 = \frac{mgL(1-\sin \theta)}{I(= mL^2 / 3)} = \frac{3g}{L} (1-\sin \theta) \]
Angular Momentum \( \mathbf{L} \)

Angular momentum \( \mathbf{L} \) is the rotary analog of linear momentum \( \mathbf{p} \equiv m\mathbf{v} \):

- definition for a (point) mass \( m \) at position \( \mathbf{R} \) (in an inertial frame):
  \[
  \mathbf{L} \equiv \mathbf{R} \times \mathbf{p} = \mathbf{R} \times m\mathbf{v}
  \]
- for a single point mass \( m \), \( \mathbf{R} \) and \( \mathbf{p} \) define a plane \((x,y)\) and \( \mathbf{L} \) is perpendicular to this plane...
  \[
  \mathbf{L} = R_p \sin \theta = \mathbf{R}_\perp = R_\perp \mathbf{p}
  \]
- For a collection of masses \( m_i \)
  the individual angular momenta \( \mathbf{L}_i \) add (vectorially):
  \[
  \mathbf{L} = \sum_i \mathbf{L}_i = \sum_i \mathbf{R}_i \times \mathbf{p}_i = \sum_i \mathbf{R}_i \times m_i \mathbf{v}_i
  \]
  
- Looking at this, our mouth starts to water:
  - we see \( m_i \mathbf{v}_i \) and we know \( \mathbf{F}_{\text{Net}} = d(m\mathbf{v})/dt \)!
- Let's try to take the time-derivative of \( \mathbf{L} \):
  \[
  \frac{d\mathbf{L}}{dt} = \frac{d}{dt} \left( \sum_i \mathbf{R}_i \times m_i \mathbf{v}_i \right) = \sum_i \left[ \left( \frac{d\mathbf{R}_i}{dt} \times m_i \mathbf{v}_i \right) + \left( \mathbf{R}_i \times m_i \frac{d\mathbf{v}_i}{dt} \right) \right]
  = \sum_i \left[ \left( \mathbf{v}_i \times m_i \mathbf{v}_i \right) + \left( \mathbf{R}_i \times \mathbf{F}_i \right) \right] = \sum_i \left[ 0 + \tau_i \right] = \sum_i \tau_i = \tau_{\text{Net}}
  \]
- Note: we have NOT required the body be rigid!
Angular Momentum Conservation

In case NO NET EXTERNAL TORQUE acts on a spinning object, ANGULAR MOMENTUM (direction AND magnitude) is CONSERVED!

\[ \text{If} \quad \frac{d\mathbf{L}}{dt} = \tau_{\text{Net}} = 0 \quad \Rightarrow \quad \mathbf{L} = \text{constant} \]

- Conservation of Angular Momentum

- This is the principle of the gyroscope: a fast-spinning top, suspended in gimbals so that no external torques can act, will keep its axis of rotation fixed in an inertial frame...

- We will see in a moment that \( \mathbf{L} = \mathbf{I} \omega \) in many instances, and if we have large \( I \) and/or large \( \omega \), also \( L \) will be large, and it will take a correspondingly large torque to change \( L \) significantly!
Angular Momentum $\mathbf{L}$, and $\omega$

Consider a mass $m$ on a bent massless rigid stick (the poor man’s rattle):

- In this example $\mathbf{L}$ rotates around the rotation axis denoted by $\omega$.
- Thus: $\mathbf{L}$ is constant in magnitude, but NOT in direction!
- Thus $d\mathbf{L}/dt \neq 0$ and an external torque is needed from the hand ...

In case the axis of rotation is a **Symmetry Axis** of the body, no external torque is required ... (balanced rotation)

- Then: $\tau_{\text{Ext}} = 0$, \quad $\mathbf{L} = (\mathbf{L}_1+\mathbf{L}_2) \parallel \omega$!
- AND: $\mathbf{L} = (\mathbf{L}_1+\mathbf{L}_2) = (\mathbf{L}_1\parallel + \mathbf{L}_2\parallel)$
  \[ = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{r}_i \omega = \sum_i m_i r_i^2 \omega = I \omega \]
Examples

A blob of putty of mass $m$ falls (and sticks) onto the rim of a horizontally mounted disk of radius $R$, spinning on fixed axis with $\omega_0$ and moment of inertia $I$.

- **Q**: express the final angular velocity in terms of the known quantities.

\[ L_i = L_f \Rightarrow I\omega_i = I_f\omega_f = (I + mR^2)\omega_f \Rightarrow \omega_f = \omega_i/(1 + mR^2/I) \]

- **Note**: $K_f = \frac{1}{2}I\omega_i^2/(1 + mR^2/I) < \frac{1}{2}I\omega_i^2 = K_i \Rightarrow \text{inelastic}$

A skater spins at 1.0 rev/s, then retracts her arms... Treat the outstretched arms as a uniform “stick” of $m=8.0 \text{ kg}$ and $L=1.8 \text{ m}$ long; when she retracts, her arms are along her body at $R=0.20 \text{ m}$. The body is modeled as a uniform cylinder of $r=0.12 \text{ m}$ radius and $M=50 \text{ kg}$.

- **Body**: $I_B = \frac{1}{2}Mr^2 = 0.36 \text{ kgm}^2$
- **Arms**: $I_{Ai} = mL^2/12 = 2.16 \text{ kgm}^2$, $I_{Af} = mR^2 = 0.32 \text{ kgm}^2$
- **Thus**: $\omega_f = \omega_i I_i/I_f = 1.0 \times (2.16+0.36)/(0.32+0.36) = 3.7 \text{ rev/s}$
Angular Momentum \( \mathbf{L} = \mathbf{R} \times \mathbf{p} = I \omega \)

- Note that any object that has momentum \( \mathbf{p} \), also has angular momentum with respect to any chosen axis...

Angular Momentum conservation:
- for the bullet \( \mathbf{L} = \text{constant} \) (if we ignore gravity): \( L = (R/2) \, mv_0 \)

Angular Momentum is conserved in the collision:
- assume the bullet stops at a distance \( R/2 \) from \( O \):

\[
L_i = L_f \quad \Rightarrow \quad L_i = |\mathbf{R}_m \times m\mathbf{v}_0| = \frac{R}{2} \, mv_0 = I \omega_f = \left( \frac{1}{2} MR^2 + m \frac{R^2}{4} \right) \omega_f
\]

- NOTE: linear momentum is NOT conserved: a large shock reaction force acts on axle!
I (100 kg) walk past a lamp post; I pass the lamp post at 2 m, with a speed of 4 m/s. My angular momentum with respect to the lamp post is …
Consider a bicycle wheel with moment of inertia \( I_{cm} = MR^2 \) (all mass in the rim) for rotation axis \( \omega \) through cm, \( \perp \) wheel, which is also a symmetry axis

1. When the wheel isn’t spinning \( (L=\omega=0) \):
   - the weight exerts net torque:
     \[
     \tau_{\text{Net}} = \mathbf{r} \times \mathbf{w} = d\mathbf{L}/dt = I_y \alpha,
     \]
     where \( I_y \) is the moment of inertia for the rotation axis through \( O, \) perpendicular to paper.

For a \textbf{FLAT} object in the \( z-y \) plane:
   - \( I_x = \sum m_i r_i^2 = \sum m_i (z_i^2 + y_i^2) = I_y + I_z \) (The \textbf{Perpendicular Axis Theorem})
     - Thus: \( I_x = MR^2 = I_y + I_z = 2I_y \),
       and \( I_y = \frac{1}{2}I_x + Mr^2 = \frac{1}{2}MR^2 + Mr^2 \)
   - And the wheel will tip down:
     \[
     \tau_{\text{Net}} = \mathbf{r} \times \mathbf{w} = d\mathbf{L}/dt = I_y \alpha_y
     \]
2. When the wheel is spinning ($\mathbf{L} = I\mathbf{\omega}$):

- the weight still exerts a net torque:
  \[ \tau_{\text{Net}} = \mathbf{r} \times \mathbf{w} = \tau_y = \frac{d\mathbf{L}}{dt}, \]
  \[ \tau_y = d\mathbf{L}/dt, \]
- where $d\mathbf{L}/dt$, as before, is directed along the $y$-axis, i.e. through $O$, into the paper.
- However, now $\mathbf{L} = \mathbf{L}_x \neq 0$, and $d\mathbf{L} = d\mathbf{L}_y$ adds to it in time $dt$, i.e. $\mathbf{L}$ will rotate around $O$, changing direction, but constant in magnitude!
- Thus: instead of tipping down, the spinning wheel executes horizontal circles around $z$ (≡ PRECESSION) with angular velocity:

\[ \omega_z = \frac{d\phi}{dt} = \frac{dL}{L} = \frac{d\mathbf{L}}{dt} = \frac{\tau_y}{L} = \frac{\tau_y}{I_x \omega_x} \]

- The faster $\omega_x$, the slower precession...
- We assumed $\omega_z \ll \omega_x$, so that $\mathbf{L} = \mathbf{L}_z + \mathbf{L}_x \approx \mathbf{L}_x$