Honors Classical Physics I

PHY141
Lecture 19
Rotational Dynamics

Please set your Clicker Channel to 21
Calculation of \( I \)

- **Wide hoop**: axis through center, \( \perp \) plane of hoop: = collection of thin hoops of radius \( r \) and mass \( dm \):

\[
I = \int_0^M r^2 \, dm = \int_{R_1}^{R_2} r^2 \left( \frac{2\pi r \, dr}{\pi(R_2^2-R_1^2)} \right) M = \frac{2M}{R_2^2-R_1^2} \int_{R_1}^{R_2} r^3 \, dr
\]

\[
= \frac{2M}{R_2^2-R_1^2} \frac{1}{4} \left[ R_2^4-R_1^4 \right] = \frac{1}{2} M \left( R_2^2 + R_1^2 \right)
\]

- and idem for a hollow cylinder on its axis...

- **Solid disk or cylinder**: on its symmetry axis:

\( R_1 = 0, R_2 = R \): \( I_{cm} = \frac{1}{2} MR^2 \)

- Note: always \( I = fML^2 \), where \( M \) is the mass of the object, \( L \) its "typical" dimension, and \( f \) some factor that depends on the object's shape and the location of the rotation axis!

  - This may seem trivial, but it provides a powerful check of your calculations!
  - Note: possibly \( f > 1 \)!
In order to calculate the moment of inertia for a solid cylinder of mass $M$ and radius $R$

A. I need to know the height $H$ of the cylinder

B. I need to know the rotation axis

C. I need to know the torque on the cylinder
Parallel Axis Theorem \( I_d = I_{//cm} + Md^2 \)

- As noted: for any extended object there exist an INFINITE number of moments of inertia \( I \): it all depends on the position and direction of the axis!

- Now, we can prove that we can relate any \( I_A \) (for an axis through \( A \)), to the \( I_{//cm} \) for a PARALLEL axis through the CM:

\[
\begin{align*}
\mathbf{r}_i &= \mathbf{d}_{CM} + \mathbf{R}_i; \text{ i.e: } x_i = d_{CM,x} + X_i, \quad y_i = d_{CM,y} + Y_i \\
I_A &\equiv \sum_i m_i r_i^2 = \sum_i m_i \left( x_i^2 + y_i^2 \right) \\
&= \sum_i m_i \left( d_{CM,x}^2 + x_i^2 + 2d_{CM,x}X_i + d_{CM,y}^2 + Y_i^2 + 2d_{CM,y}Y_i \right) \\
&= \sum_i m_i d_{CM}^2 + \sum_i m_i R_i^2 + 2d_{CM,x} \sum_i m_i X_i + 2d_{CM,y} \sum_i m_i Y_i = Md_{CM}^2 + I_{//cm} + 0 + 0
\end{align*}
\]

for CM, in CM coordinate system: \( 0 = \mathbf{R}_{CM} \equiv \sum_i \frac{m_i \mathbf{R}_i}{M} \)
Application of //Axis Theorem

- **NOTE:** \( I = \int r^2 dm = \sum m_i r_i^2 \)
  
  - **Stick;** axis through center, and end, \( \perp \) stick:
    
    ![Stick Diagram]

    \[
    I_{cm} = \int_0^M r^2 dm = \int_{-L/2}^{+L/2} x^2 \left( \frac{dx}{L} M \right) = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx = \frac{1}{12} ML^2
    \]
    
    - **Check:** \( I_{end} = I_{cm} + Md^2 = \frac{1}{12} ML^2 + M \left( \frac{L}{2} \right)^2 = \frac{1}{3} ML^2 \)

  - **Uniform Disk;** axis \( \perp \) disk, through point on the rim:
    
    ![Disk Diagram]

    \[
    I_{rim} = I_{cm} + Md^2 = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2
    \]
Examples

Round objects rolling down a slope...

- 2 round objects, with the same radius $R$, roll down from rest an incline of angle $30^\circ$, and total length $L$.
  
  object 1: a **uniform solid cylinder**; object 2: a **thin hollow cylinder**. Rolling is w/o slipping and coefficient of rolling friction is zero.

- Q: calculate the **total kinetic energy** of each, and the **time** for each to reach the bottom of the slope (which one is fastest?)

- $\Delta E = 0; \quad \Rightarrow \Delta U_G = Mgh = \Delta K = \frac{1}{2} I_{CM} \omega_{CM}^2 + \frac{1}{2} Mv_{CM}^2$

  - in all cases we have that $v_{cm} = v_{rim} = \omega R$, thus $\omega = v_{CM}/R$
  
  Thus: $Mgh = \frac{1}{2} I_{CM} \omega_{CM}^2 + \frac{1}{2} Mv_{CM}^2 = \frac{1}{2} \left( I_{CM}/R^2 \right) v_{CM}^2 + \frac{1}{2} Mv_{CM}^2$

  - solving for $v_{CM}^2 = 2gh \sqrt{\left(1 + \frac{I_{CM}}{MR^2}\right)}$

  - using the constant-a formulae one finds: $t = L/v_{avg} = 2L/v_{CM} = …$

  - Because $\frac{I_{hollow}}{M_{hollow} R^2} = 1 > \frac{I_{solid}}{M_{solid} R^2} = \frac{1}{2}$

  we have: $t_{hollow} = \sqrt{\frac{2}{3/2}}$, $t_{solid} = \sqrt{\frac{4}{3}} t_{solid} > t_{solid}$
Dynamics: Back to Torque $\tau$

Torque is a force times its Arm, i.e. distance of shortest approach between the force and the axis of rotation...

- Consider the force $F$ on a mass $m$ allowed to rotate around a fixed axis $A$ at radius $R$:

\[
F_{\text{rad}} = F \cos \theta
\]

\[
F_{\tan} = F \sin \theta
\]

- The Torque of $F$ with respect to Axis $A$ ($\perp$ Paper): $\tau \equiv RF_{\tan} = RF \sin \theta$

- if $F_{\tan} = F_{\text{Net}}$ the RESULT of this torque is:

\[
\tau_{\text{Net}} \equiv RF_{\text{Net}} = RF_{\tan} = R(ma) = R(maR) = mR^2 \alpha = I \alpha
\]

- for a SYSTEM of masses $m_i$ and torques $\tau_i$: $\tau_{\text{Net}} = \Sigma_i \tau_i = \Sigma_i m_i r_i^2 \alpha = I \alpha$

where $\tau_{\text{Net}}$ simply equals the net torque by EXTERNAL forces (internal forces, and thus their torques, cancel because $F_{ij} = -F_{ji}$.)
Vector Character of $\tau$

We are used to thinking about vector forces...

In deriving the Torque equation $\tau_{\text{net}} = \sum \tau_i = I \alpha$ we used the vector equation $\mathbf{F}_{\text{net}} = m\mathbf{a}$ as our underlying Law, so where did the vector character go???

- Consider again:
  $\tau \equiv RF_{\tan} = RF \sin \theta$

- Torque clearly has a direction ... best represented by the axis of rotation!

- Torque is the product of two vectors $\mathbf{R}$ and $\mathbf{F}$ and proportional to the sine of their included angle $\theta_{\mathbf{R,F}}$, giving rise to a third vector $\tau$ perpendicular to both $\mathbf{R}$ and $\mathbf{F}$ (i.e. parallel to the rotation axis): $\tau = \mathbf{R} \times \mathbf{F}$

New definition: VECTOR (or “CROSS”) product:

- Vector Product magnitude: $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ with $|\mathbf{C}| = AB \sin \theta_{\mathbf{AB}}$
- Direction: RIGHT-HAND-RULE:
  - right-hand palm along $\mathbf{A}$, curl fingers along $\mathbf{B}$, then outstretched thumb is direction of $\mathbf{C}$!
Working with Cross Products

Calculate the torque of a force \( F = 2i + 2j \) acting at a distance \( R = –1i + 2j \) from the rotation point:

\[
\tau = \mathbf{R} \times \mathbf{F} = \mathbf{i} \left( R_y F_z - R_z F_y \right) + \mathbf{j} \left( R_z F_x - R_x F_z \right) + \mathbf{k} \left( R_x F_y - R_y F_x \right)
\]

\[
= \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    R_x & R_y & R_z \\
    F_x & F_y & F_z \\
  \end{vmatrix} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    -1 & 2 & 0 \\
    2 & 2 & 0 \\
  \end{vmatrix}
\]

\[
\equiv \mathbf{i} \left( (2\times0) - (0\times2) \right) - \mathbf{j} \left( (-1\times0) - (0\times2) \right) + \mathbf{k} \left( (-1\times2) - (2\times2) \right)
\]

\[
= 0\mathbf{i} + 0\mathbf{j} - 6\mathbf{k}
\]

Note: the cross product \( \mathbf{A} \times \mathbf{B} \) equals the AREA of the parallelogram spanned by vectors \( \mathbf{A}, \mathbf{B} \)
The Result of a Net Torque

• The net torque $\tau$ is simply the VECTOR sum of all the torques by EXTERNAL forces exerted on a body (the internal forces cancel in the vector sum because $F_{ij} = -F_{ji}$).

We derive from $F_{\text{net}} = ma$:

$$|\tau_{\text{net}}| = \left| \sum_i \tau_i \right| = \left| \sum_i \mathbf{r}_i \times \mathbf{F}_i \right| = \sum_i r_i F_{\text{tan},i} = \left( \sum_i m_i r_i^2 \right) \alpha = I \alpha$$

• where we seemed to have “lost” the vector character ...

• However, we can assign a direction to $\alpha$ (and idem to $\omega$ and $\Theta$!): parallel to the torque!

• Thus we find our $F_{\text{net}} = ma$ equivalent for rotations:

$$\tau_{\text{net}} = I \alpha$$
Summary

Moment of Inertia $I$:
- definition: $I = \int r^2 dm = \sum m_i r_i^2$ (note units kgm$^2$)
- depends on ROTATION AXIS!!
- Parallel Axis Theorem: $I_d = I_{// \text{cm}} + Md^2$

Kinetic Energy:
- for rotation about a “fixed axis” (axis fixed in an inertial system):
  $K = \frac{1}{2}I\omega^2$
- when rotating AND moving with cm axis' direction fixed:
  $K = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega_{\text{cm}}^2$

Gravitational Potential Energy of Extended body:
- sea level gravity: $U_G = Mgh_{\text{cm}}$ i.e. as if all mass $M$ were in the cm

Torque vector:
- Net Torque: $\tau_{\text{Net}} = \mathbf{R} \times \mathbf{F} = I\alpha$
- Cross Product:
  \[
  \tau = \mathbf{R} \times \mathbf{F} \equiv \mathbf{i}\left(R_y F_z - R_z F_y\right) + \mathbf{j}\left(R_z F_x - R_x F_z\right) + \mathbf{k}\left(R_x F_y - R_y F_x\right) = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  R_x & R_y & R_z \\
  F_x & F_y & F_z
  \end{vmatrix}
  \]
  Note the circular sequence $x, y, z, x, ...$