Honors Classical Physics I

PHY141
Lecture 18
Rotational Dynamics

Please set your Clicker Channel to 21
Dynamics: Torque and Work

Torque is a Force times its Arm, i.e. distance of shortest approach between the force and the axis of rotation...

- Consider the “effect” of a force on a bolt when using a wrench:

\[
F_{\text{rad}} = F \cos \theta \\
F_{\text{tan}} = F \sin \theta
\]

The Torque of \( F \) with respect to Axis \( A \) (\( \perp \) Paper, through the axis of the bolt):

\[
\tau \equiv \mathbf{R} \times \mathbf{F} ; \quad \text{magnitude: } \tau = RF_{\tan} = RF \sin \theta
\]

- Note that \( F_{\text{rad}} \) is balanced by the reaction force from bolt onto the wrench
- \( F_{\text{rad}} \) does NO WORK, b/c the motion of the end of the wrench is tangential!
- \( F_{\text{tan}} \) is the ONLY effective component!
to disengage a tight bolt easiest, given a fixed amount of force, …

A. pull perpendicular to the wrench handle
B. use a longer wrench
C. both A and B
A force of 8.0 N pulls at a distance of 0.50 m from a rotation axis. The angle between the distance vector and the force direction is 150 degrees; the torque’s magnitude is ...
Work and Rotational Kinetic Energy

• The work done by a torque \( \tau \) in moving the wrench through an angle \( \phi \) equals the force integrated over the distance:

\[
W = \int F \cdot ds = \int F_{\text{tan}} ds
\]

\[
= \int F_{\text{tan}} (Rd\phi) = \int RF_{\text{tan}} d\phi = \left. \int \tau d\phi \right|_0^\phi = \tau \phi
\]

- If \( F \) is the NET FORCE, then this is the total WORK, and MUST RESULT in a CHANGE IN KINETIC ENERGY of the wrench plus bolt!

\[
F_{\text{rad}} = F\cos\theta
\]

\[
F_{\text{tan}} = F\sin\theta
\]

\[
W = \int \tau d\phi = \Delta K, \text{ with }
\]

\[
K = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i \left( \omega r_i \right)^2
\]

\[
= \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2
\]

- where \( I \equiv \sum_i m_i r_i^2 = \int r^2 dm \) is defined as the "Moment of Inertia" \( I \)

- \( I \) describes how the mass of the rotating object is distributed around the rotation axis!
Kinetic Energy

For extended objects that move “freely”, generally their kinetic energy can be split in two parts:

- the kinetic energy of the Center-of-Mass (CM) MOTION,
- plus the kinetic energy of ROTATION AROUND a CM AXIS!

• Proof: the coordinate $\mathbf{r}_i$ of point $i$ on a body can be written in all cases as the vector sum $\mathbf{r}_i = \mathbf{R}_{cm} + \mathbf{R}_i$, where $\mathbf{R}_{cm}$ is the coordinate of the cm, and $\mathbf{R}_i$ is the coordinate of $i$ in the cm coordinate system:

• differentiating w.r.t. time $t$ gives for any such point $i$:

$$\mathbf{v}_i = \mathbf{V}_{CM} + \mathbf{V}_i,$$

and thus:

$$K \equiv \frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} \sum_i m_i \left( V_{CM,x}^2 + V_{i,x}^2 + 2 V_{CM,x} V_{i,x} + V_{CM,y}^2 + V_{i,y}^2 + 2 V_{CM,y} V_{i,y} \right)$$

$$= \frac{1}{2} \sum_i m_i \left( V_{CM}^2 + V_i^2 \right) + V_{CM,x} \sum_i m_i V_{i,x} + V_{CM,y} \sum_i m_i V_{i,y}$$

$$= \frac{1}{2} \sum_i m_i V_{CM}^2 + \frac{1}{2} \sum_i m_i V_i^2 + 0 + 0$$

$$V_i = \omega_{CM} \mathbf{R}_i$$

$$= \frac{1}{2} \sum_i m_i V_{CM}^2 + \frac{1}{2} I_{CM} \omega_{CM}^2,$$

$I_{CM} \equiv \sum_i m_i \mathbf{R}_i^2$

for CM, in CM coordinate system: $0 = \mathbf{R}_{CM} = \frac{\sum_i m_i \mathbf{R}_i}{M}$
Calculation of Moment of Inertia $I$

- **DEPENDS ON AXIS DIRECTION & POSITION!!!** $I = \int r^2 dm = \sum m_i r_i^2$

  - **Dumbbell**: axis through center, ⊥ halter:
    \[ I = mR^2 + MR^2 = (m + M)R^2 \]
    - Note dimensions!

  - **Stick**: axis through center, ⊥ stick:
    \[ I = \int_0^L r^2 dm = \int_{-L/2}^{+L/2} x^2 \left( \frac{dx}{M} \right) = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx \]
    \[ = \frac{1}{3} \frac{M}{L} \left[ \frac{L^3}{8} - \left( -\frac{L^3}{8} \right) \right] = \frac{1}{12} ML^2 \]
    - Note: idem for “door” of width $L$ !!

  - **Stick**: axis through end, ⊥ stick:
    \[ I = \int_0^L r^2 dm = \int_0^L x^2 \left( \frac{dx}{LM} \right) = \frac{M}{L} \int_0^L x^2 dx = \frac{1}{3} ML^2 \]
    - and idem for door with axis at hinges...

  - **Hoop**: axis through center, ⊥ plane of hoop:
    \[ I = \int_0^M r^2 dm = R^2 \int_0^M dm = MR^2 \]
    - idem for thin cylinder on axis...
Calculation of $I$

- **Wide hoop**: axis through center, $\perp$ plane of hoop: = collection of thin hoops of radius $r$ and mass $dm$:

$$I = \int_0^M r^2 dm = \int_{R_1}^{R_2} r^2 \left( \frac{2\pi r \, dr}{\pi(R_2^2 - R_1^2)} \right) M = \frac{2M}{R_2^2 - R_1^2} \int_{R_1}^{R_2} r^3 dr$$

$$= \frac{2M}{R_2^2 - R_1^2} \frac{1}{4} \left[ R_2^4 - R_1^4 \right] = \frac{1}{2} M \left( R_2^2 + R_1^2 \right)$$

- and idem for a **hollow cylinder** on its axis...

- **Solid disk or cylinder**: on its symmetry axis:

  $R_1 = 0, R_2 = R$: $I_{cm} = \frac{1}{2} MR^2$

- **Note**: always $I = fML^2$, where $M$ is the mass of the object, $L$ its "typical" dimension, and $f$ some factor that depends on the object’s shape and the location of the rotation axis!

  - This may seem trivial, but it provides a powerful check of your calculations!
  - Note: possibly $f > 1$!