Honors Classical Physics I

PHY141
Lecture 11
Work and Kinetic Energy

Please set your Clicker Channel to 21
Orbits and Central Forces

• **Ellipse Geometry:**

In cartesian \((x, y)\) coordinates: \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\):

\[r = r_0 \frac{1+e}{1+e \cos \theta}\]

\[a = r_0 \frac{1}{1-e} ; \quad b = r_0 \sqrt{\frac{1+e}{1-e}} \]

\[r_0 = a - \sqrt{a^2 - b^2} , \quad e = \sqrt{1 - \frac{b^2}{a^2}}\]

Transformation between parameters:

Note: from ellipse to other geometries: \(r = r_0 \frac{1+e}{1+e \cos \theta}\):

\[e = 0 \Rightarrow \text{circle, radius } r_0\]

\[0 < e < 1 \Rightarrow \text{ellipse}\]

\[e = 1 \Rightarrow \text{parabola}\]

\[e > 1 \Rightarrow \text{hyperbola}\]
Orbits and Central Forces (2)

- Derivation of Orbits:

Generally: \( \mathbf{F} = f_r \hat{r} + f_\theta \hat{\theta} \); \( f_\theta = 0 \) for a central force!

\[
\text{dr} = dr \hat{r} + r d\theta \hat{\theta} \quad \Rightarrow \quad \mathbf{v} = \frac{d\mathbf{s}}{dt} = \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta} \equiv \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}
\]

using dot notation for \( \frac{d}{dt} \)

\[
\Rightarrow \mathbf{a} = \frac{dv}{dt} = \dot{v} = \dot{r} \hat{r} + \dot{r} \frac{dr}{dt} + r \dot{\theta} \hat{\theta} + r \ddot{\theta} \frac{d\theta}{dt} = \dot{r} \hat{r} + r \ddot{\theta} \hat{\theta} + r \dot{\theta} \hat{\theta} + r \dddot{\theta} + r \dot{\theta} (-\hat{r} \hat{\theta})
\]

\[
= (\ddot{r} - r \dddot{\theta}^2) \hat{r} + (2r \ddot{\theta} + r \dddot{\theta}) \hat{\theta}
\]

\( \mathbf{F} = m \mathbf{a} \) and central: \( f_r / m = \ddot{r} - r \dddot{\theta}^2 \) and \( f_\theta / m = 2r \ddot{\theta} + r \dddot{\theta} = \frac{d}{dt} \left( r^2 \dot{\theta} \right) = 0 \)

Therefore \( r^2 \dot{\theta} = \ell \) is a constant (of the motion) ! (Angular Momentum conservation)
Orbits and Central Forces (3)

- **Central force** → *Angular Momentum is conserved*

We found for a central force: \( f_r/m = \ddot{r} - r \dot{\theta}^2 \) and \( \ell = r^2 \dot{\theta} = \text{constant} \).

Now, we will use the new variable \( u = r^{-1} \) and eliminate the \( t \)-dependence to find the orbit; we use: \[
\dot{r} = \frac{d(u^{-1})}{dt} = -\frac{u}{u^2} \text{ chain rule } = -r^2 \frac{du}{d\theta} \frac{\dot{\theta}}{\theta} = -\ell \frac{du}{d\theta}
\]

Continuing: \[
\ddot{r} = -\ell \frac{d}{dt} \left( \frac{du}{d\theta} \right) = -\ell \frac{d^2u}{d\theta^2} \frac{\dot{\theta}}{\theta} = -\ell u^2 \frac{d^2u}{d\theta^2}; \text{ similarly: } \dot{\theta}^2 = \ell^2 r^4 = u^4 \ell^2
\]

\[
\Rightarrow f_u/m = \ddot{r} - r \dot{\theta}^2 = -\ell^2 u^2 \frac{d^2u}{d\theta^2} - \ell^2 u^3 \quad \Rightarrow \quad \frac{d^2u}{d\theta^2} + u = \frac{-f_u}{m\ell^2 u^2}
\]

The solution \( u(\theta) \) of this complicated differential equation is the orbit!
Orbits and Central Forces (4)

- Orbits

We found for a central force:
\[
\frac{d^2u}{d\theta^2} + u = \frac{-f_u}{ml^2u^2}
\]

Now: the orbit for a gravitational (or Coulomb) force: \( f_u = -ku^2 \)

The differential equation becomes:
\[
\frac{d^2u}{d\theta^2} + u = \frac{k}{ml^2} \quad \text{with solution:}
\]

\[
u(\theta) = \frac{k}{ml^2} + A\cos(\theta + \theta_0) \quad \text{(easy to check; } A \text{ and } \theta_0 \text{ are 'arbitrary' parameters} \}
\]

Going back to \( r(\theta) \):
\[
r(\theta) = u^{-1} = \frac{1}{k/\ml^2 + A\cos(\theta + \theta_0)} = \frac{m\ell^2/k}{1 + (Am\ell^2/k)\cos(\theta + \theta_0)}
\]

Comparing this with the equation for an ellipse in polar coordinates, we find:
\[
r(\theta) = r_0 \frac{1+e}{1+e\cos\theta}, \quad \text{with eccentricity } e = Am\ell^2/k \text{ and perigee } r_0 = \frac{m\ell^2/k}{1+e} = \left(\frac{k}{ml^2} + A\right)^{-1}
\]
Recap

• Newton's Laws:
  - 1 & 2: \( \mathbf{F}_{\text{Net}} = m\mathbf{a} \)
    - LHS: Sum of forces acting ON the object with mass \( m \)
    - RHS: the resulting acceleration, from which the motion can be deduced
  - 3: \( \mathbf{F}_{AB} = -\mathbf{F}_{BA} \) (Action is Reaction)
  - Valid only in Inertial Reference Frames!
  - Free Body Diagram to find the Net Force \( \mathbf{F}_{\text{Net}} \)
    - diagram with all the (vector) forces acting ON the body of MASS \( m \)
    - Show your choice of axes clearly!
    - If appropriate: make separate diagrams for separate sub-problems...

• Friction (example of a common force):
  - Friction \( \mathbf{F}_f \approx \mu \mathbf{N} \), where the coefficient \( \mu = \mu_s \) of static friction is typically larger than \( \mu = \mu_k \) of kinetic (= moving) friction...
  - The direction of friction is opposing the motion...
  - The normal force \( \mathbf{N} \) presses the surfaces together; it is normal, i.e. perpendicular, to the surfaces

• Uniform Circular Motion \( \Rightarrow a_c = \frac{v^2}{R} \)
Work by a Constant Force

• What matters about a force?
  - its EFFECT: i.e. does it lead to some useful result, a displacement?
  - We called friction a nuisance; what do we mean by that? It takes away from our applied effort...

• In physics, WORK (the useful result of applying a force) has a precisely defined meaning...

• A particular force may do work on an object if there is a displacement along (a component of) the force’s direction:
  - pushing a refrigerator from one corner of the kitchen to another
  - lifting boxes with printer paper on a shelf
  - walking up the stairs

• For a constant vector force \( F \) and vector displacement \( s \), work \( W \) is defined as the dot-product (scalar product):

\[
W = F \cdot s = Fs \cos \theta_{F,s}
\]

(constant \( F, s \))

Units: Joule = [J] \( \equiv \) [N m]

\( 9/19/2014 \)

Lecture 11
Examples

- Sliding that refrigerator at constant speed over a horizontal distance $D$: assume we push horizontally
- with constant force $F$:
  - Free Body Diagram:

  ![Free Body Diagram](image)

  Note: All constant forces

  - Work by $F$ (me!): $W_F = F \cdot D = FD\cos0^\circ = FD$
  - Work by Gravity: $W_G = W \cdot D = WD\cos90^\circ = 0$
  - Work by Friction ($F_f = -F$): $W_f = F_f \cdot D = F_fD\cos180^\circ = -F_fD$
  - Work by the Normal force ($N = -W$): $W_N = N \cdot D = ND\cos90^\circ = 0$

9/19/2014
Example

Sliding a crate of mass $M$ up a ramp from floor onto bench of height $h$:

Q: what is work done by the various forces?

- Normal force: $N \perp D \Rightarrow W_N = 0$
- Pushing/Pulling force $F//D$: $W_F = FD$
- Friction:
  $W_f = -F_f D$;
  $(\cos \theta = \cos 180^\circ = -1)$
- Gravity:
  $W_G = F_G \cdot D$
  $= F_G D \cos (90^\circ + \theta)$
  $= -F_G D \sin \theta = -F_G h$

- Thus: in this process gravity is doing -ve work in the amount of $Mgh$!
- It is just the vertical distance (in the direction of the force) that counts, and does not depend on the slope or the precise shape of the incline!!

Similar with a crate sliding down an incline...
A crate is moved by me up a ramp; select the answer that is correct

A. Gravity is doing negative work on the crate...
B. I am doing negative work on the crate...
C. The normal force is doing negative work on the crate...
D. No positive work is done on the crate at all...
A crate is moved by me up a ramp; select the answer that is correct

A. Gravity is doing positive work on the crate...
B. I am doing positive work on the crate...
C. The normal force is doing positive work on the crate...
D. No negative work is done on the crate at all...

B.

9/19/2014 Lecture 11

94%
Work by Varying Forces

- Work is a **scalar quantity**, and we can thus “add little pieces of work” together, i.e. sum a series of dot-products of the force $F(s_i)$ with small parts $ds_i$ of the total trajectory...

- Thus, **the general definition of Work**:

$$W \equiv \lim_{n \to \infty} \sum_{i=1}^{n} F_i \cdot ds_i = \int_{s_0}^{s_1} F(s) \cdot ds$$

- i.e. for $F(s)$ constant:

$$W \equiv \int_{s_0}^{s_1} F(s) \cdot ds = F \cdot \int_{s_0}^{s_1} ds = F \cdot (s_1 - s_0) = F \cdot \Delta s = F \Delta s \cos \theta_{F,s}$$
Total Work and Kinetic Energy

• Typically the result of the total work on a mass \(m\) is not only displacement, but often also a change in its velocity.
  - This - of course - follows from Newton's Law \(F_{\text{Net}} = ma\)
  - Let's look in more detail:

\[
F_{\text{Net}} = \sum_i F_i = ma \equiv m \frac{dv}{dt}
\]

\[
W_{\text{tot}} = \sum_i W_i = \sum_i \int_{s_0}^{s_1} F_i \cdot ds = \int_{s_0}^{s_1} \sum_i F_i \cdot ds = \int_{s_0}^{s_1} F_{\text{Net}} \cdot ds
\]

\[
= \int_{s_0}^{s_1} ma \cdot ds = m \int_{s_0}^{s_1} \frac{dv}{dt} \cdot ds = m \int_{s_0}^{s_1} \frac{ds}{dt} \cdot dv = m \int_{v_0}^{v_1} v \cdot dv
\]

\[
= m \frac{1}{2} (v_1^2 - v_0^2) = \frac{1}{2} mv_1^2 - \frac{1}{2} mv_0^2 \equiv K_1 - K_0 = \Delta K
\]

• where we defined a new SCALAR: Kinetic Energy \(K \equiv \frac{1}{2} mv^2\), so that TOTAL WORK done ON a object with mass \(m\), moving it from position initial to final, equals the CHANGE in ITS KINETIC ENERGY \(\Delta K \equiv K_f - K_i\).
**Child going down a Slide**

- a child is going down a slide of vertical height $h$, starting from rest; calculate the work by Gravity and the child’s $v_f$

\[
W_G = \int w \cdot ds = mg \int ds \cos \theta = mg \int dy = mgh
\]

- *If friction is absent*, this is ALL the work that is done on the child (the Normal Force does NOT contribute; is always \( \perp \) to displacement $ds$!). Then:

\[
W_{tot} = mgh = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - 0(\text{rest}) \quad \text{thus} \quad v_f = \sqrt{2gh}
\]

- i.e. we found $v_f$ *without knowing the detailed shape of the slide*, something quite impossible using our early kinematics!
A crate is moved by me up a ramp (from rest to a stop at the top); select the answer that is correct:

A. The total work done on the crate is zero ...
B. The total work done on the crate is positive ...
C. The total work done on the crate is negative ...

See previous slides:

\[ W_{net} = W_G + W_f + W_{me} = \Delta K = 0 \]
Work by Friction

- If friction cannot be ignored, the previous problem becomes very much more complicated; we have to know the detailed shape of the slide, because the frictional force (and thus its total work) depends on the normal force, which in turn varies with the local angle of the slide...

  - Work by friction, because it always opposes any motion, is always negative! If friction is present in the previous problem, the final speed $v_f$ will be less because the total work done on the child will be less...

  - Work by Normal forces is always zero, because they always act $\perp$ to the displacement!
Example

A block is kicked up an incline of angle $\theta$ at initial speed $v_0$. The coefficient of kinetic friction is $\mu_k$. Calculate the distance $D$ that the block will slide up if it returns with half the initial speed:

- Make a diagram with axes, labels, and known (and unknown) quantities!
- Total work = work by friction only (work by gravity = 0 at return!)

\[
W_f = -\mu_k mg 2D \cos \theta = \frac{1}{2} mv_0^2 \left( \frac{1}{4} - 1 \right) \Rightarrow D = \ldots
\]

\[
\text{Work by friction is negative...}
\]

- Total Work = Change in Kinetic Energy:

\[
W_f = -\mu_k mg 2D \cos \theta = \Delta K = \frac{1}{2} mv_0^2 \left( \frac{1}{4} - 1 \right) \Rightarrow D = \ldots
\]

- This would be MUCH more lengthy/difficult to solve using force considerations...