Honors Classical Physics I

PHY141
Lecture 3
Motion in Two and Three Dimensions
2-Dimensional Motion

Motion is often in 2 or more dimensions.
- You need 2 (or 3) axes in your sketch! Label them carefully.

Choose a perpendicular axis-system so as to simplify the calculus:
- e.g. the y-axis along the vertical in case of simple gravity problems,
- and the x-axis along one constant direction of the motion ...
- or axes parallel and perpendicular to a sloping incline, or ...

The velocity and the motion (velocity, acceleration) are sums of vectors:
- e.g. for constant acceleration: \( \mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t \); \( \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0t + \frac{1}{2}\mathbf{a}t^2 \)
  - (NOTE: a constant vector is constant in magnitude and in direction!)

[  Remember Vector sum: if \( \mathbf{C} = \mathbf{A} + \mathbf{B} \) then: \( C_x = A_x + B_x \), and \( C_y = A_y + B_y \), etc.]
- therefore: \( v_x(t) = v_{0x} + a_xt \); \( x(t) = x_0 + v_{0x}t + \frac{1}{2}a_xt^2 \)

\[ v_y(t) = v_{0y} + a_yt \; ; \; y(t) = y_0 + v_{0y}t + \frac{1}{2}a_yt^2 \]

\( \ldots \)

[  the motion in each direction \((x, y)\) is independent and a separate 1-dimensional motion problem!]
2-Dimensional Motion - Projectile Motion

Motion under influence of near-Earth gravity and **ABSENCE of DRAG**; i.e. **CONSTANT** acceleration of \( g = 9.80 \text{ m/s}^2 \) downwards: \( \mathbf{a} = -g\mathbf{j} \) (with \( y \)-axis pointing up). Generally a 3-dimensional problem ...

**Choose axes** to simplify the calculus:

- the \( y \)-axis along the **vertical** (anti-parallel to gravitational acceleration!)
  - this automatically ensures that the **only** acceleration (in absence of drag) is in \( y \)-direction: \( a_y = -g \); **while** \( a_x = a_z = 0 \)

- the \( x \)-axis along the horizontal component of \( \mathbf{v}_0 \):
  - i.e. \( v_{0x} = v_0 \cos \phi \) and \( v_{0y} = v_0 \sin \phi \)
    \( (\phi \equiv \text{angle between horizon and } \mathbf{v}_0) \)
  - Thus \( v_z=0 \) and \( a_z=0 \); this ensures that there will be no motion along \( z \)!

- the motion is therefore **TWO-dimensional** in this situation!
  - the equations are:
    \[
    \begin{align*}
    v_x(t) &= v_{0x} + a_x t = v_{0x} ; \\
    x(t) &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 = x_0 + v_{0x} t \\
    v_y(t) &= v_{0y} + a_y t = v_{0y} - gt ; \\
    y(t) &= y_0 + v_{0y} t + \frac{1}{2} a_y t^2 = y_0 + v_{0y} t - \frac{1}{2} gt^2
    \end{align*}
    \]

- (Generally (e.g. when wind and drag cannot be ignored) the motion is **THREE-dimensional** ...)

08/29/2014
A ball is shot with speed $v_0=20 \text{ m/s}$ at angle $\phi=30^\circ$ up from a cliff of height $h=12 \text{ m}$.

- **Q1**: Calculate the shot’s horizontal range $R$:
- **Sketch**:

- **Discussion**: Constant acceleration: $a_y=-g$, $a_x=0$

- **Calculation** (keep it in symbols!):

  $x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2 = 0 + (v_0 \cos \phi) t + 0$

  $y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2 = h + (v_0 \sin \phi) t - \frac{1}{2}gt^2$

Briefly consider this for a moment:

- with $v_0$, $\phi$ known, we can calculate $x(t)$ and $y(t)$ at ANY given time $t$; i.e. we know exactly where the projectile is (the “motion”) as function of $t$...
- Because $a_x=0$, we have $x$ simply proportional to $t$, and: $t = x / (v_0 \cos \phi) = x / v_{0x}$ ...
- Between the TWO equations we may eliminate $t$, and find $y(x)$: the TRAJECTORY:

  $y = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_{0y} \left( \frac{x}{v_{0x}} \right) - \frac{g}{2v_{0x}^2} x^2$

  note: $y(x=R)=0$, which allows to solve for $R$!
for projectile motion at sea level and without drag, select the correct statement(s):

A. the horizontal component of the motion is linear with time
B. the horizontal component of the motion depends only on the horizontal component of the initial velocity
C. the vertical component of the motion depends only on the vertical component of the initial velocity

(assume the starting point is \(x_0=0\) and \(y_0=0\), and \(g=9.80\ \text{m/s}^2\))

A. A only
B. B only
C. C only
D. A & B but not C
E. A & C but not B
F. B & C but not A
G. A & B & C (all are true)
**Q1: Calculate the shot's horizontal range $R$:**

- **Sketch:**
  
  - $v_0 = 20 \text{ m/s}$
  - $\phi = 30^\circ \text{ up}$
  - $h = 12 \text{ m}$

- **Calculation of TIME when $y=0$ (landing): this depends ONLY on $y$!!**

  \[
y = h + v_0 \sin \phi \cdot t - \frac{1}{2}gt^2 = 0
  \]

  \[
  \Rightarrow t_{1,2} = \frac{v_0 \sin \phi \pm \sqrt{v_0^2 \sin^2 \phi + 2gh}}{-g}
  \]

  \[
  \Rightarrow t_{1,2} = \frac{10 \pm \sqrt{10^2 + 2 \times 9.80 \times 12}}{9.80} = \frac{10 \pm 18.3}{9.80} = -0.85 \text{ s}, 2.89 \text{ s}
  \]

- **TWO times are found because a parabola like the one here intersects the $x$-axis at two points**
  - the *math* doesn’t know that $t>0$; for $x<0$ we expect $t<0$!
  - But WE know: $t=2.89 \text{ s}$ is the *physical* solution.

- **Then we use the $x$-equation of motion to find the horizontal RANGE $R$ at $t=t_2$:**

  \[
  R = x(t=t_2) = v_{0x}t_2 = (20 \text{ m/s}) \cos 30^\circ \times 2.89 \text{ s} = 50.0 \text{ m}
  \]

\[2.9 \text{ s makes sense!}\]
Example (cont’d)

- **Q2**: Calculate the TRAJECTORY, i.e. \( y(x) \), and the range \( R \):

- **Sketch**:

- **Calculation of TRAJECTORY**: eliminate \( t \) between the \( x \)- and the \( y \)-equations of motion:

\[
x = v_{ox} t \quad \Rightarrow \quad t = x/v_{ox}
\]

\[
y = h + v_{oy} t - \frac{1}{2} g t^2 = h + \frac{v_{oy}}{v_{ox}} x - \frac{g}{2v_{ox}^2} x^2 = y(x)
\]

⇒ hits ground: \( y(x=R) = 0 \):

\[
0 = h + \frac{v_{oy}}{v_{ox}} R - \frac{g}{2v_{ox}^2} R^2 \quad \Rightarrow \quad R_{1,2} = \frac{-\frac{v_{oy}}{v_{ox}} \pm \sqrt{\left(\frac{v_{oy}}{v_{ox}}\right)^2 + \frac{2gh}{v_{ox}^2}}}{g/v_{ox}} = \frac{v_{oy} \pm \sqrt{v_{oy}^2 + 2gh}}{g/v_{ox}} = -7.3 \text{ m}, \ 50.0 \text{ m}
\]

- clearly, TWO distances are found (the math doesn’t distinguish), but \( x=50 \text{ m} \) is the correct answer!
- **Example (cont’d)**

- **Q3**: Calculate the maximum height of the trajectory:

- **Sketch**: 

- Many ways of doing this; e.g.: 
  - calculate the $y$ when $dy/dx=0$ (point of flat slope): 

  $$y = h + v_{0y} t - \frac{1}{2} g t^2 = h + \frac{v_{0y}}{v_{0x}} x - \frac{g}{2v_{0x}^2} x^2 = y(x)$$

  $$\Rightarrow \frac{dy}{dx}\bigg|_{x=x_m} = 0 = \frac{v_{0y}}{v_{0x}} - \frac{g}{2v_{0x}^2} x_m \Rightarrow x_m = \frac{v_{0y} v_{0x}}{g}$$

  $$\Rightarrow y_m = h + \frac{v_{0y}^2}{g} - \frac{v_{0y}^2}{2g} = h + \frac{v_{0y}^2}{2g} = 17.1 \text{ m}$$

- another (simpler) way: find the point where $v_y=0$:

  $$v_y = v_{0y} - gt = 0 \Rightarrow t_m = \frac{v_{0y}}{g}$$

  $$y_m = h + v_{0y} t_m - \frac{1}{2} g t_m^2 = h + \frac{v_{0y}^2}{g} - \frac{v_{0y}^2}{2g} = h + \frac{v_{0y}^2}{2g} = 17.1 \text{ m}$$

- $v_0=20 \text{ m/s}$
- $\phi=30^\circ \text{ up}$
- $h=12 \text{ m}$
Example (cont’d)

- **Q4**: Calculate the initial ANGLE that gives the maximum range:
  - \( \phi \) is now a variable!

- Find the range as function of \( \phi \):
  - **First (simplest!), for** \( h = 0 \):
    
    \[
    R = R_2 = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2gh}}{g/v_{0x}} = \frac{2v_{0x}v_{0y}}{g} = \frac{2v_0 \cos \phi \cdot v_0 \sin \phi}{g} = \frac{v_0^2 \sin \phi \cos \phi}{g} = \frac{v_0^2 \sin 2\phi}{g}
    \]

    this is maximum for \( \sin 2\phi = 1 \), i.e. \( \phi = 45^\circ \)

  - **For** \( h \neq 0 \):
    
    \[
    R = R_2 = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2gh}}{g/v_{0x}} = \frac{v_{0x}v_{0y} \left( 1 + \sqrt{1 + 2gh/v_{0y}^2} \right)}{g} = \frac{v_0^2 \sin(2\phi) \left( 1 + \sqrt{1 + 2gh/(v_0 \sin \phi)^2} \right)}{2g} = R(\phi)
    \]

    Then: \( \frac{dR}{d\phi} = 0 \) for a maximum; thus, a very complicated equation!
Recap

- **Motion:**
  - the displacement or position $r$ of an object as function of time: $r(t)$ (vector!)
  - the study of motion led Newton to the break-through in mechanics, the *universal* law:
    $$ F = m \ a $$

- **Pure Definitions:** (i.e. NO derivations!)
  
  Motion $\equiv r(t)$;  
  $$ r(t) = x(t)i + y(t)j + z(t)k $$

  Velocity (vector):
  
  $$ v(t) \equiv \frac{dr(t)}{dt} = \frac{dx(t)}{dt}i + \frac{dy(t)}{dt}j + \frac{dz(t)}{dt}k $$  
  
  $$ = v_x i + v_y j + v_z k $$

  Acceleration (vector):
  
  $$ a(t) \equiv \frac{dv(t)}{dt} = \frac{dv_x(t)}{dt}i + \frac{dv_y(t)}{dt}j + \frac{dv_z(t)}{dt}k $$  
  
  $$ = a_x i + a_y j + a_z k $$
Motion in Two or Three Dimensions

• All quantities \( x(t), v(t), a(t) \) now become 2 (or 3-) dimensional VECTORS!
  - Note: we’ve tacitly kept the notation general already before, when we discussed one-dimensional motion...

• Given \( a_x(t), v_{0x}, x_0 \) and \( a_y(t), v_{0y}, y_0 \), we can independently and separately calculate \( x(t) \) and \( y(t) \) (and \( z(t) \))
  - as if they are independent one-dimensional motions!
  - Thus, we can reconstruct the full 2 or 3-dimensional motion!
  - Moreover (in case \( z(t)=0 \)), using \( x(t) \) and \( y(t) \) we may calculate the TRAJECTORY \( y(x) \) by eliminating the time parameter \( t \)
Example: A Tree hit by an Arrow

A arrow is shot at an angle of $\phi = 45^\circ$ up and hits a tree at $R = 220$ m away, at the same height as its starting point.

- **Q1**: How long was the arrow’s flight (the “time-to-hit”) $t_H$?
- **Sketch**:
  - Constant acceleration:
    \[ a_y = -g, \ ax = 0 \]
- **Calculation**:
  (keep it in symbols!):
  \[
  \begin{align*}
  x &= x_0 + v_{0x}t + \frac{1}{2}a_xt^2 = 0 + v_{0x}t + 0 \\
  \Rightarrow \quad x_H &= R = v_{0x}t_H
  \\
  y &= y_0 + v_{0y}t + \frac{1}{2}a_yt^2 = 0 + v_{0y}t - \frac{1}{2}gt^2 \\
  \Rightarrow \quad y_H &= 0 = v_{0y}t_H - \frac{1}{2}gt_H^2
  
  \end{align*}
  \]

  Two equations, three unknowns $(v_{0x}, v_{0y}, t_H)$

  - However, with $\phi$ known, we have $v_{0x} = v_0 \cos \phi$ and $v_{0y} = v_0 \sin \phi$, thus two of the unknowns $\Rightarrow$ one: $v_0$!

  - In particular: here $\phi = 45^\circ$, so that: $v_{0x} = v_{0y}$

  - Between the TWO equations we may **eliminate** $v_{0x}$ & $v_{0y}$, and find $t_H$:
  \[
  0 = v_{0y}t_H - \frac{1}{2}gt_H^2 = R \frac{v_{0y}}{v_{0x}} - \frac{1}{2}gt_H^2 \quad \Rightarrow \quad t_H = \sqrt{\frac{2R \tan \phi}{g}} \quad \text{ (tan}\phi = 1 \text{ for } \phi = 45^\circ)
  \]

  Or: using previous slide: $R = \frac{v_0^2 \sin 2\phi}{g}$, find $v_0 \Rightarrow v_{0y}$, and then $t_H = \frac{2v_{0y}}{g}$
Summary Kinematics

• Definitions of $v$, $a$:

  - Motion: $\mathbf{r}(t)$;  
    Velocity (vector): $\mathbf{v}(t) \equiv \frac{d\mathbf{r}(t)}{dt} \quad \Leftrightarrow \quad \mathbf{r}(t) - \mathbf{r}_0 = \int_0^t \mathbf{v} dt$

  - Acceleration (vector): $\mathbf{a}(t) \equiv \frac{d\mathbf{v}(t)}{dt} \quad \Leftrightarrow \quad \mathbf{v}(t) - \mathbf{v}_0 = \int_0^t \mathbf{a} dt$

• Special case: **constant acceleration**:
  - Motion: $\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a} t$, $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2 \quad \Rightarrow \quad \mathbf{v}^2 - \mathbf{v}_0^2 = 2 \mathbf{a} \cdot (\mathbf{r} - \mathbf{r}_0)$

• Projectile motion: trajectory is a parabola (if no drag)
  - Trajectory: $y(x) = \ldots$
    - $x = v_{0x} t \quad \Rightarrow \quad t = x/v_{0x}$
    - $y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 = y_0 + \frac{v_{0y}}{v_{0x}} x + \frac{a_y}{2v_{0x}^2} x^2 = y(x)$
    - $R(\text{range}) = v_0^2 \sin 2\varphi/g$ (when the starting and landing points are at equal height!)

08/29/2014