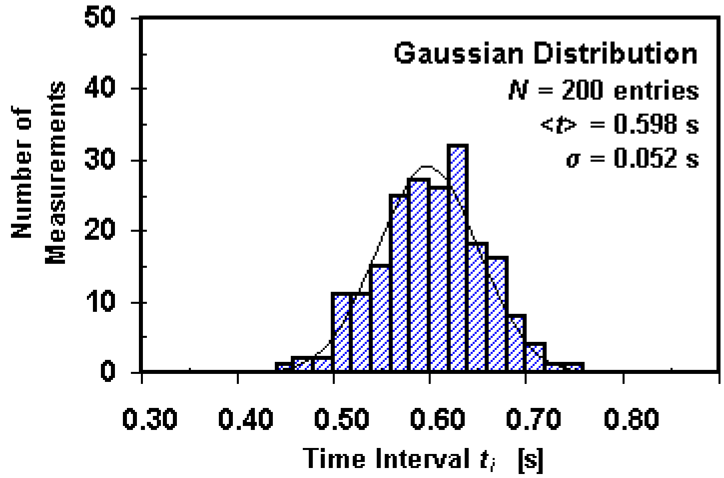
PHY133 - Classical Physics I Laboratory — Error Analysis

Statistical or Random Errors

Every measurement an experimenter makes is uncertain to some degree. The uncertainties are of two kinds: (1) random errors, or (2) systematic errors. For example, in measuring the time required for a weight to fall to the floor, a random error will occur when an experimenter attempts to push a button that starts a timer simultaneously with the release of the weight. If this random error dominates the fall time measurement, then if we repeat the measurement many times (*N* times) and plot equal intervals (bins) of the fall time t_i on the horizontal axis against the number of times a given fall time t_i occurs on the vertical axis, our results (see histogram below) should approach an ideal bell-shaped curve (called a Gaussian distribution) as the number of measurements *N* becomes very large.



The best estimate of the *true* fall time *t* is the *mean* value (or average value) of the distribution $\langle t \rangle$:

$$\langle t \rangle = (\Sigma^{N}_{i=1}\ t_i)/N\ . \qquad (1)$$

If the experimenter squares each deviation from the mean, averages the squares, and takes the square root of that average, the result is a quantity called the "root-mean-square" or the "standard deviation" s of the distribution. It measures the random error or the statistical uncertainty of the individual measurement t_i:

$$s = \sqrt{[\Sigma^{N}_{i=1}(t_i - \langle t \rangle)^2 / (N\text{-}1)\ ]}. \qquad (2)$$

About two-thirds of all the measurements have a deviation less than one s from the mean and 95% of all measurements are within two s of the mean. In accord with our intuition that the *uncertainty of the mean* should be smaller than the uncertainty of any single measurement, measurement theory shows that in the case of random errors the standard deviation of the mean, s_m, is given by:

$$s_m = s / \sqrt{N} ,$$

where N again is the number of measurements used to determine the mean. Then the result of the N measurements of the fall time would be quoted as $t = \langle t \rangle \pm s_m$.

Whenever you make a measurement that is repeated N times, you are supposed to calculate the mean value and its standard deviation as just described. For a large number of measurements this procedure is somewhat tedious. If you have a calculator with statistical functions it may do the job for you. There is also a simplified prescription for estimating the random error which you can use. Assume you have measured the fall time about ten times. In this case it is reasonable to assume that the largest measurement t_{max} is approximately $+2s$ from the mean, and the smallest t_{min} is $-2s$ from the mean. Hence:

$$s \approx ¼ (t_{max} - t_{min})$$

is an reasonable estimate of the uncertainty in a single measurement. The above method of determining s is a rule of thumb if you make of order ten individual measurements (i.e. more than 1 and less than 100).

Uncertainty due to Instrumental Precision

Not all errors are statistical in nature. That means some measurements cannot be improved by repeating them many times. For example, assume you are supposed to measure the length of an object (or the weight of an object). The accuracy will be given by the spacing of the tickmarks on the measurement apparatus (the meter stick). You can read off whether the length of the object lines up with a tickmark or falls in between two tickmarks, but you could not determine the value to a precision of l/10 of a tickmark distance. Typically, the error of such a measurement is equal to one half of the smallest subdivision given on the measuring device. So, if you have a meter stick with tickmarks every mm (millimeter), you can measure a length with it to an accuracy of about 0.5 mm. While in principle you could repeat the measurement numerous times, this would not improve the accuracy of your measurement!

Note: This assumes of course that you have not been sloppy in your measurement but made a careful attempt to line up one end of the object with the zero of the meter stick as accurately as you can, and that you read off the other end of the meter stick with the same care. If you want to judge how careful you have been, it would be useful to ask your lab partner to make the same measurements, using the same meter stick, and then compare the results.

Systematic Errors

Systematic errors result when characteristics of the system we are examining, or the instruments we use are different from what we assume them to be. For example, if a voltmeter we are using was calibrated incorrectly and reads 5% higher than it should, then every voltage reading we record using this meter will have an error of 5%. Clearly, taking the average of many readings will not help us to reduce the size of this systematic error. If we knew the size and direction of the systematic error we could correct for it and thus eliminate its effects completely. Even when we are unsure about the effects of a systematic error we can sometimes estimate its size (though not its direction) from knowledge of the quality of the instrument. For example, the meter manufacturer may guarantee that the calibration is correct to within 1%. (Of course, one pays more for an instrument that is guaranteed to have a small error.)

Propagation of Errors

Even simple experiments usually call for the measurement of more than one quantity. The experimenter inserts these measured values into a prescribed formula to compute a desired result. He/she will want to know the uncertainty of the result. Fortunately, many real situations can be treated in one of the ways discussed below. If you are faced with a more complex situation, ask your lab instructor for help.

Additive Formulae

When a result R is calculated from two measurements x and y, with uncertainties Δx and Δy, and two constants a and b with the additive formula:

$$R = ax + by\ ,$$

and if the errors in x and y are independent, then the error in the result R will be:

$$(\Delta R)^2 = (a\,\Delta x)^2 + (b\,\Delta y)^2\ .$$

The reason why we should use this quadratic form and not simply add the uncertainties $a\Delta x$ and $b\Delta y$, is that we don't know whether x and y were both measured too large or too small; indeed the measurement errors on x and y might cancel each other in the result R! Independent errors cancel each other with some probability (say you have measured x somewhat too big and y somewhat too small; the error in R might be small in this case). This partial statistical cancellation is correctly accounted for by adding the uncertainties quadratically.

Note: a and b can be positive or negative, i.e. the equation works for both addition and subtraction.

Multiplicative Formulae

When the result R is calculated by multiplying a constant a times a measurement of x times a measurement of y (or divided by y), i.e.:

$$R = axy \quad \text{or} \quad R = ax/y,$$

then the *relative* errors $\Delta x/x$ and $\Delta y/y$ add quadratically:

$$(\Delta R/R)^2 = (\Delta x/x)^2 + (\Delta y/y)^2\ .$$

Example: Say quantity x is measured to be 1.00, with an uncertainty $\Delta x = 0.10$, and quantity y is measured to be 1.50 with uncertainty $\Delta y = 0.30$, and the constant $a = 5.00$. The result R is obtained as $R = 5.00 \times 1.00 \times 1.50 = 7.5$. The relative uncertainty in x is $\Delta x/x = 0.10$ or 10%, whereas the relative uncertainty in y is $\Delta y/y = 0.20$ or 20%. Therefore the relative error in the result is $\Delta R/R = \sqrt{(0.10^2 + 0.20^2)} = 0.22$ or 22%,. The absolute uncertainty of the result R is obtained by multiplying 0.22 with the value of R: $\Delta R = 0.22 \times 7.50 = 1.7$.

More Complicated Formulae

If your result is obtained using a more complicated formula, as for example:

$$R = a\,x^2 \sin y\ ,$$

there is a very easy way to find out how your result R is affected by errors Δx and Δy in x and y. Insert into the equation for R, instead of the value of x, the value $x+\Delta x$, and find how much R changes:

$$R + \Delta R_x = a\,(x+\Delta x)^2 \sin y \,.$$

If y has no error you are done. If y has an error as well, do the same as you just did for x, i.e. insert into the equation for R the value for $y+Dy$ instead of y, to obtain the error contribution ΔR_y. The total error of the result R is again obtained by adding the errors due to x and y quadratically:

$$(\Delta R)^2 = (\Delta R_x)^2 + (\Delta R_y)^2 \,.$$

This way to determine the error always works and you could use it also for simple additive or multiplicative formulae as discussed earlier. Also, if the result R depends on yet another variable z, simply extend the formulae above with a third term dependent on Δz.

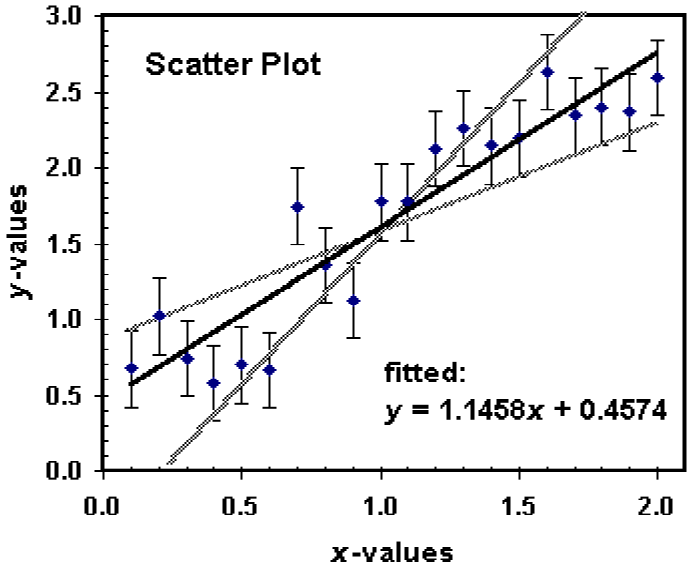
Significant Figures

In light of the above discussion of error analysis, discussions of significant figures (which you should have had in previous courses) can be seen to simply imply that an experimenter should quote digits which are appropriate to the uncertainty in his result. The above result of $R = 7.5 \pm 1.7$ illustrates this. It would not be meaningful to quote R as 7.53142 since the error affects already the first figure. On the other hand, to state that $R = 8 \pm 2$ is somewhat too casual. It is a good rule to give one more significant figure after the first figure affected by the error.

Fitting a Straight Line through a Series of Points

Frequently in the laboratory you will have the situation that you perform a series of measurements of a quantity y at different values of x, and when you plot the measured values of y versus x you observe a linear relationship of the type $y = ax + b$. Your task is now to determine, from the errors in x and y, the uncertainty in the measured slope a and the intercept b. There is a mathematical procedure to do this, called "linear regression" or "least-squares fit". We will instead use a much simpler, graphical method, briefly described in the following.

Plot the measured points (x,y) and mark for each point the errors Δx and Δy as bars that extend from the plotted point in the x and y directions. Draw the line that best describes the measured points (i.e. the line that minimizes the sum of the squared distances from the line to the points to be fitted; the least-squares line). This line will give you the best value for slope a and intercept b. Next, draw the steepest and flattest straight lines, see the Figure, still consistent with the measured error bars. From these two lines you can obtain the largest and smallest values of a and b still consistent with the data, a_{min} and b_{min}, a_{max} and b_{max}. From their deviation from the best values you then determine, as indicated in the beginning, the uncertainties Δa and Δb.



Further Reading

- Introductory: J.R. Taylor, *An Introduction to Error Analysis*, Oxford UP, 1982.
- Advanced: R. Bevington and D.K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences*, 2nd. edition, McGraw-Hill NY, 1992.

June 1992