Classical Physics II

PHY132
Lecture 25
Electromagnetic Radiation
Maxwell’s Equations

Four equations together completely describe all the material covered so far (i.e. all of electromagnetism); they are called Maxwell’s Equations:

on the direct sources of E and B:
- Gauss' Law for E: \[ \oint E \cdot dA = Q_{\text{encl}} / \varepsilon_0 \] (our workhorse for the E-field)
- Gauss' Law for B: \[ \oint B \cdot dA = 0 \] (no magnetic monopoles)

on the deep connections between E and B:
- Ampere’s Law: \[ \oint B \cdot dI = \mu_0 \left( i_{\text{charge}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{\text{encl.}} \] (with Maxwell’s \( i_D \))
- Faraday’s Law: \[ \mathcal{E} = \oint E \cdot dI = -\frac{d\Phi_B}{dt} \] (with Lenz’ Law)

  • note: symmetry would be perfect if magnetic monopoles and currents of monopoles existed ...

Maxwell showed that these 4 laws predict the existence of ELECTROMAGNETIC waves, propagating through vacuum, at the speed of light \( c = 1 / \sqrt{\varepsilon_0 \mu_0} \)

- This was revolutionary at the time: waves were only thought to propagate in a MEDIUM ...
- Maxwell postulated the AETHER as the medium for EM waves... with very weird properties ...
Thinking a bit about waves…

• Consider a charge which is “shaking back-and-forth”.
• Such a charge is being continuously accelerated...
• The E-field lines of a static charge are thus “shaken back and forth” as well and the field wiggles propagate outwards with the speed of light;
• The moving charge also creates magnetic field circles, which oscillate in strength:
  • a time-varying magnetic field is created at any point, perpendicular to the local E-field.
• The time-varying B-field, in turn, sustains a time-varying E-field at any location, perpendicular to itself…
Electromagnetic Waves

Combining:
\[
\frac{\partial E_y(x,t)}{\partial x} = -\frac{\partial B_z(x,t)}{\partial t} \quad \Rightarrow \quad \frac{\partial^2 E_y(x,t)}{\partial x^2} = -\frac{\partial^2 B_z(x,t)}{\partial x \partial t}
\]

\[-\frac{\partial B_z(x,t)}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y(x,t)}{\partial t} \quad \Rightarrow \quad -\frac{\partial^2 B_z(x,t)}{\partial t \partial x} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y(x,t)}{\partial t^2}\]

\[
\frac{\partial^2 E_y(x,t)}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y(x,t)}{\partial t^2} \quad \Rightarrow \quad E_y(x,t) = E_{\text{max}} \sin(kx - \omega t)
\]

\[
\nu = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299,792,458 \text{ m/s} \equiv c
\]

- i.e. the electromagnetic radiation spreads with speed of light \( c \)!

Similarly: \( B_z(x,t) = B_{\text{max}} \sin(kx - \omega t) \)

- The propagation direction is \( \perp \) to both \( E \) and \( B \), which are themselves mutually perpendicular! Direction from right hand rule: \( E \times B \)

- filling in \( E_y \) and \( B_z \) in the 1st equation above:

\[
\frac{\partial E_y(x,t)}{\partial x} = kE_y = -\frac{\partial B_z(x,t)}{\partial t} = -(-\omega)B_z \quad \Rightarrow \quad E_y = \frac{\omega}{k} B_z = cB_z
\]

Note: any repetitive function \( f(kx - \omega t) \) will be a solution if
\[
\omega/k = 2\pi/T \lambda/(2\pi) = \lambda/T = \lambda f = \nu = c.
\]

Indeed, any repetitive function \( f(kx - \omega t) \) can be built from a sum of sine and cosine waves with frequencies and wavelengths such that \( \nu = \lambda f \) (Fourier theorem)
The Generation of EM Waves

Oscillating charges/currents (e.g. in an antenna) will generate and emit electromagnetic radiation...

- they lose energy in the process, which needs to be replenished (radio emitters need energy)
- depending on the shape of the antenna, the emitted radiation is uni-directional (a plane wave) or omni-directional (spherical wave), or anything in between...
- Note: waves may be emitted in many different directions : e.g. a vertical dipole antenna has a mostly horizontal pattern of emission
**E and B Field in a Laser Beam**

A laser emits (infrared) light with $\lambda = 9.8 \, \mu m$ along $x$; its electric field amplitude $E_{\text{max}} = 1.2 \, \text{MV/m}$ and polarized along $y$ express the electric and magnetic fields as function of position $x$ and time $t$.

- **first, find $\omega$ and $k$**: 
  \[ \omega \equiv \frac{2\pi}{T} = 2\pi f = 2\pi \frac{c}{\lambda} = 1.9 \times 10^{14} \, \text{rad/s} \]
  \[ k \equiv \frac{2\pi}{\lambda} = \frac{\omega}{c} = 6.4 \times 10^5 \, \text{m}^{-1} \]

- **Find the B-field amplitude**: 
  \[ B_{\text{max}} = E_{\text{max}} / c = 4.0 \times 10^{-3} \, \text{T} \]

- **Combining**: 
  \[
  E = (1.2 \times 10^6 \, \text{V/m})\, \mathbf{j} \sin \left( 6.4 \times 10^5 x - 1.9 \times 10^{14} t \right)
  
  B = (4.0 \times 10^{-3} \, \text{T})\, \mathbf{k} \sin \left( 6.4 \times 10^5 x - 1.9 \times 10^{14} t \right)
  
- i.e. $B_{\text{max}}$ is typically small for electromagnetic radiation!
So $E$ dominates optical interactions

- Index of refraction is due to the dielectric response of materials to electromagnetic waves.
  - Glass is a strong, and transparent, dielectric!
- With very high intensity light beams (e.g., femtosecond laser pulses), one can ionize atoms due to the electric field of the light beam.
Energy Flow

Energy density in an electromagnetic wave:

\[ u = u_E + u_B = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} = \frac{\varepsilon_0 E^2}{2} + \frac{E^2}{2c^2 \mu_0} = \varepsilon_0 E^2 \]

- i.e. the energy densities of the electric and magnetic fields are equal!

the ENERGY FLOW \( S \) is defined as:
- the amount of energy transported per unit time (i.e. \( \text{POWER} \)),
- per unit area in the direction of travel...

i.e. take the energy \( dU \) in a thin volume \( Adx \), with \( A // x \), the direction of travel:

\[ dU = uA dx = uA c dt \]

\[ S = \frac{P}{A} = \frac{dU}{Adt} = uc = \varepsilon_0 c E^2 = \varepsilon_0 c^2 EB = \frac{EB}{\mu_0} \]

- defining the Energy Flow VECTOR (Poynting Vector) \( S \):

\[ S = \frac{E \times B}{\mu_0} \]

and:

\[ S = \frac{EB}{\mu_0} = \frac{E_{\max} B_{\max} \cos^2(\ldots)}{\mu_0} = \frac{E_{\max} B_{\max}}{2\mu_0} \]

- e.g. the laser’s Power/Area is:

\[ S = \frac{(1.2 \times 10^6 \ \text{V/m})(4.0 \times 10^{-3} \ \text{T})/8\pi \times 10^{-7} \ \text{T \cdot m/A}}{1.9 \times 10^9 \ \text{W/m}^2} = 1.9 \ \text{W/mm}^2 \]
Momentum Flow

The Momentum flow RATE (i.e. \(dp/dt\)) of the EM wave per unit transverse area is then:

\[
P = F_v = \frac{dp}{dt} \frac{c}{c}; \quad \frac{P}{A} = S; \quad \Rightarrow \frac{dp}{dt} = \frac{P}{c} = \frac{SA}{c} \quad \Rightarrow \quad \frac{1}{A} \frac{dp_{EM}}{dt} = \frac{S}{c}
\]

- If this momentum rate flow is fully absorbed, there will be a resulting pressure (i.e. average force per unit area), called radiation pressure \(P_{rad}\):

\[
P_{rad, abs} = \frac{F}{A} = \frac{1}{A} \frac{dp}{dt} = \frac{1}{A} \frac{dp_{EM}}{dt} = \frac{S}{c}
\]

- If, in contrast, the wave is fully reflected, the momentum-rate-of-change is twice as big because \(\Delta p = p_{refl} - p_{inc} = -2p_{inc}\), and:

\[
P_{rad, refl} = \frac{F}{A} = \frac{1}{A} \frac{dp}{dt} = \frac{2}{A} \frac{dp_{EM}}{dt} = \frac{2S}{c}
\]
Example: Solar Sails

Consider the radiation pressure on a $A = 1.0 \text{ km}^2$ solar sail constructed out of ultra-thin $t = 10 \mu\text{m}$ thick fully reflecting Mylar. Take the density of Mylar equal to $\rho = 1000 \text{ kg/m}^3$. The solar radiation at Earth orbit is $S_{\text{avg}} = 1.5 \text{ kW/m}^2$.

Calculate the acceleration $a$ ...

$$P_{\text{rad, refl}} = \frac{F}{A} = \frac{ma}{A} = \frac{\rho Ata}{A} = \rho ta = \frac{2S}{c}$$

$$\Rightarrow a = \frac{2S}{\rho tc} = \frac{3.0 \times 10^3}{1000 \times 10 \times 10^{-6} \times 3 \times 10^8} = 1 \text{ mm/s}^2$$

- Thus, a very small but steady acceleration,
  - $v = 3.6 \text{ m/s after 1 hr}$;
  - diminishing when you go further out in the solar system ...
- Any payload would further reduce the acceleration, of course...