Classical Physics II

PHY132
Lecture 23
Maxwell’s Equations
Maxwell’s Equations

Four equations together completely describe all the material covered so far (i.e. all of electromagnetism); they are called Maxwell’s Equations:

on the direct sources of E and B:
- Gauss’ Law for E: \( \int \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \) (our workhorse for the E-field)
- Gauss’ Law for B: \( \oint \mathbf{B} \cdot d\mathbf{A} = 0 \) (no magnetic monopoles)

on the deep connections between E and B:
- Ampere’s Law: \( \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( i_{\text{Charge}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{\text{encl.}} \) (with Maxwell’s \( i_D \))
- Faraday’s Law: \( \varepsilon = \oint \mathbf{E} \cdot d\mathbf{l} = - \frac{d\Phi_B}{dt} \) (with Lenz’ Law)

\[ \text{\textbullet note: symmetry would be perfect if magnetic monopoles and currents of monopoles existed ...} \]

Maxwell showed that these 4 laws predict the existence of ELECTROMAGNETIC waves, propagating through vacuum, at the speed of light \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \)

- This was revolutionary at the time: waves were only thought to propagate in a MEDIUM ...
- Maxwell postulated the AETHER as the medium for EM waves... with very weird properties ...
Thinking a bit about waves...

- Consider a charge which is “shaking back-and-forth”.
- Such a charge is being continuously accelerated...
- The E-field lines of a static charge are thus “shaken back and forth” as well and the field wiggles propagate outwards with the speed of light;
- The moving charge also creates magnetic field circles, which oscillate in strength:
  - a time-varying magnetic field is created at any point, perpendicular to the local E-field.
  - The time-varying B-field, in turn, sustains a time-varying E-field at any location, perpendicular to itself...
E and B perpendicular to v

Take the x-axis along v. Consider a “thin” Gaussian box aligned with x, y, z:

- the transverse components $E_y$ and $E_z$ cancel in the field integral over the surface of the box:

- $E_y$ and $E_z$ go IN on one side and OUT on the opposite side:

$$\oint E_y \, dA_y = E_y \, dA_y - E_y \, dA_y = Q_{\text{enclosed}} / \varepsilon_0 = 0$$

- Choosing the far side of the box at large x (where the wave hasn’t come yet: $E=0$),

- it follows from Gauss’ law $\oint \mathbf{E} \cdot d\mathbf{A} = Q_{\text{enclosed}} / \varepsilon_0$

  that $E_x=0$ everywhere ($Q_{\text{enclosed}}=0$)

Similarly, $\oint \mathbf{B} \cdot d\mathbf{A} = 0$ proves that $B_y$ is zero...

i.e. both $\mathbf{E}$ and $\mathbf{B}$ are perpendicular each other, and to v!
Consider the time-varying $\Phi_E$ (i.e. displacement current $i_D$):

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{\text{encl.}} \]

\[ \oint \mathbf{B} \cdot d\mathbf{l} = -a_{ef} B_z(x+\Delta x, t) + 0_{fg} + a_{gh} B_z(x, t) + 0_{he} \]

\[ = -a \left( B_z(x+\Delta x, t) - B_z(x, t) \right) \]

\[ = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \frac{\partial E_y(x, t)}{\partial t} a\Delta x \quad \text{(Ampere)} \]

\[ \Rightarrow \frac{\partial B_z(x, t)}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y(x, t)}{\partial t} \]
Electromagnetic Waves

Now, consider the time varying $\Phi_B$:

$$\mathcal{E} \equiv \int_{efgh} \mathbf{E} \cdot d\mathbf{l} = a_{ef} E_y(x+\Delta x, t) + 0_{fg} - a_{gh} E_y(x, t) + 0_{he}$$

$$= a\left( E_y(x+\Delta x, t) - E_y(x, t) \right)$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{\partial B_z(x, t)}{\partial t} a\Delta x \text{ (Faraday)}$$

$$= a\left( E_y(x+\Delta x, t) - E_y(x, t) \right)$$

$$\Rightarrow \frac{\partial E_y(x, t)}{\partial x} = -\frac{\partial B_z(x, t)}{\partial t}$$
Electromagnetic Waves

Combining: \[ \frac{\partial E_y(x,t)}{\partial x} = -\frac{\partial B_z(x,t)}{\partial t} \Rightarrow \frac{\partial^2 E_y(x,t)}{\partial x^2} = -\frac{\partial^2 B_z(x,t)}{\partial x \partial t} \]

\[ -\frac{\partial B_z(x,t)}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y(x,t)}{\partial t} \Rightarrow -\frac{\partial^2 B_z(x,t)}{\partial t \partial x} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y(x,t)}{\partial t^2} \]

\[ \frac{\partial^2 E_y(x,t)}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y(x,t)}{\partial t^2} \Rightarrow E_y(x,t) = E_{\text{max}} \sin(kx - \omega t) \]

\[ \nu = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299792458 \text{ m/s} \equiv c \]

Note: any repetitive function \( f(kx - \omega t) \) will be a solution as long as \( \omega/k = 2\pi/T \lambda/(2\pi) = \lambda/T = \lambda f = \nu = c \).

Indeed, any repetitive function \( f(kx - \omega t) \) can be built from a sum of sine and cosine waves with frequencies and wavelengths such that \( \nu=\lambda f \) (Fourier theorem)

- i.e. the electromagnetic radiation spreads with speed of light \( c \)!

Similarly: \( B_z(x,t) = B_{\text{max}} \sin(kx - \omega t) \)

- The propagation direction is \( \perp \) to both \( E \) and \( B \), which are themselves mutually perpendicular! Direction from right hand rule: \( E \times B \)

- filling in \( E_y \) and \( B_z \) in the 1\(^{\text{st}}\) equation above:

\[ \frac{\partial E_y(x,t)}{\partial x} = kE_y = -\frac{\partial B_z(x,t)}{\partial t} = -(-\omega)B_z \Rightarrow E_y = \frac{\omega}{k} B_z = cB_z \]