Classical Physics II

PHY132
Lecture 19
Mutual and Self Inductance
Changing the Angle between A, B

Consider a static and uniform B-field to the right. A wire loop (N windings, area A) rotates around an axis perpendicular to B (perpendicular to the paper), with constant angular velocity \( \omega \):

- Thus only the relative orientation of B and A varies with time...
- The change in flux:

\[
\frac{d\Phi_B}{dt} = \frac{d}{dt} (B \cdot A) = BA \frac{d}{dt} \cos \theta_{AB} = BA \frac{d}{dt} \cos(\omega t) = -\omega BA \sin(\omega t)
\]

\[
\Rightarrow \mathcal{E} = -N \frac{d\Phi_B}{dt} = \omega NBA \sin(\omega t)
\]

- i.e. induced EMF varies with time sinusoidally with amplitude \( \omega NBA \)
- Numerical example: an AC generator:

\[
f = 60 \text{ Hz} = \omega/2\pi, \quad N = 500, \quad A = 3.0 \text{ m}^2, \quad B = 0.10 \text{ T}
\]

\[
\Rightarrow \mathcal{E} = 2\pi f NBA = 120\pi \times 500 \times 3.0 \times 0.10 = 57 \text{ kV},
\]

i.e. good for High Voltage power line distribution
Inductance: EMF and Sources of B

A *changing* magnetic flux induces an EMF: a changing flux $d\Phi_{B2}/dt$ over wire loop 2 creates a current in the loop, which *opposes* the flux change. A magnetic field might be created by a current (e.g. in a solenoid), and a *changing* current $i_1$ creates a changing magnetic field $\mathbf{B}$.

$$N_1 \frac{di_1}{dt} \rightarrow \frac{dB_1}{dt} \rightarrow N_2 \frac{d\Phi_{B12}}{dt} \rightarrow \mathcal{E}_2$$

- The proportionality factor between $di_1/dt$ and $\mathcal{E}_2$ is the **mutual inductance** $M_{12}$ between the coils: $\mathcal{E}_2 = -M_{12} \frac{di_1}{dt}$ (Note the "symbolic" $-$ sign!)

- $M$ specifies the effect ($\mathcal{E}$) in a second conductor due a current change in the first, mediated by $\mathbf{B}_1$.

- Typically, $M_{12}$ depends **only** on geometrical factors
  - (size, relative orientation, #windings) of the coils...

- Unit of Inductance: $\text{H(enry)} \equiv \text{V/(A/s)} = \Omega \cdot \text{s}$
Example: the Tesla Coil

The "Tesla Coil" is a system of two "tightly coupled" coils, typically used to create very high voltages by induction...

\[ B = \begin{cases} \mu_0 N_1 i_1 / l_1 \text{ inside } A_1 \text{ near the center of the long solenoid} \\ \approx 0 \text{ outside } A_1 \text{ (idealized long solenoid)} \end{cases} \]

\[ \Rightarrow E_2 = -N_2 \frac{d\Phi}{dt} B_2 = -N_2 \frac{d}{dt} \int_{A_2} B \cdot dA \]

\[ = -N_2 \left( \frac{dB}{dt} \right) (BA_1) = -N_2 A_1 \frac{dB}{dt} = -N_2 A_1 \frac{\mu_0 N_1}{l_1} \frac{di_1}{dt} = -M_{12} \frac{di_1}{dt} \]

\[ \Rightarrow M_{12} = \frac{\mu_0 A_1 N_1 N_2}{l_1} \]

- E.g.:

- \[ N_1 / l_1 = 50000 / \text{m}; \ N_2 = 100, \ A_1 = 10 \text{ cm}^2; \]

- \[ M_{12} = 4\pi \times 10^{-7} \times 10 \times 10^{-4} \times 5 \times 10^6 = 6.3 \times 10^{-3} \text{H} = 6.3 \text{ mH} \]

- For \( i = (10 \text{ A}) \sin(10^5 t) \) (\( \omega = 10^5 \) i.e. \( f \approx 16 \text{ kHz} \)):

- \[ |E_2| = M_{12} \frac{di_1}{dt} = 6.3 \text{ mH} \times (10^6 \text{ A/s}) \cos(10^5 t) = (6.3 \text{ kV}) \cos(10^5 t) ! \]
Transformers

Transformers are used in everywhere, e.g. where AC voltages must be transformed from one value to another...

- a transformer consists of a pair of TIGHTLY COUPLED coils, such that the flux generated by the PRIMARY coil is completely coupled to the SECONDARY.

- This is best achieved with a ~circular soft-iron core on which both coils are wound: the magnetic field is then almost completely kept in the iron: The flux-per-turn \( \Phi_B \) is thus the same in both coils

\[
|\mathcal{E}_1| = V_1 = N_1 \frac{d\Phi_B}{dt} \\
|\mathcal{E}_2| = V_2 = N_2 \frac{d\Phi_B}{dt}
\]

\[
\frac{V_1}{N_1} = \frac{V_2}{N_2}
\]

- If no power loss in the transformer, then \( P = I_1 V_1 = I_2 V_2 \); 
- and if the secondary sees \( R_{\text{load}} \), the primary sees: 

\[
R_1 = \frac{V_1}{I_1} = \frac{V_1}{I_1} \frac{V_1}{V_1} = \frac{V_2^2}{I_2} \frac{N_1^2}{N_2^2} = \frac{V_2}{I_2} \frac{N_1^2}{N_2^2} = R_{\text{load}} \frac{N_1^2}{N_2^2}
\]

ideal transformer: \( R_{\text{wind}} = 0, \Phi_B = \Phi_B^2 \)
A step-up transformer must supply $14 \text{kV}_{\text{rms}}$ for a neon advertisement sign, using regular 60 Hz, $120 \text{ V}_{\text{rms}}$ household power

- calculate the required turns ratio $N_2/N_1$:

$$\frac{d\Phi_B}{dt} = \frac{V_1}{N_1} = \frac{V_2}{N_2} \quad \Rightarrow \quad \frac{N_2}{N_1} = \frac{V_2}{V_1} = \frac{14 \times 10^3}{120} = 117$$

- the secondary is to supply $8.0 \text{ mA}$, what is the primary current? assuming no power loss:

$$P_1 = I_1 V_1 = P_2 = I_2 V_2 \quad \Rightarrow \quad I_1 = I_2 \times \frac{V_2}{V_1} = 933 \text{ mA}$$

- Thus, a 1.0 A fuse in the primary circuit may prevent accidents

- The primary's wire thickness must be able to carry this current
  
  \( \Rightarrow \) cross section is \((933/8) = 120\times\) larger (thickness \(11\times\) more)
Summary: Mutual Inductance

Relates a changing current in a coil, loop or wire in general, to the induced emf in a second coil, loop or wire...

• **Unit H;** 1 Henry = 1 V/(A/s) = 1 Ω·s
• Depends on geometrical quantities only (typically): coil dimensions, number of windings, relative distance and orientation...
• **Mutual Inductance is the principle behind TRANSFORMERS**
• The magnetic field, and thus the mutual inductance can be increased tremendously, \( \times 10^5 \) or so, with the use of ferromagnetic cores (for high frequency: ferrites)
• Mutual inductance is a **nuisance** in many electronic circuits, and careful circuit design is necessary to minimize it (e.g. distancing coils and mounting them mutually perpendicular...)
• **Mutual inductance is indeed mutual, i.e.:** \( M_{12} = M_{21} \)!
  - this can be easily shown for the examples above ...
Self Inductance

Whereas Mutual Inductance describes the effect of a changing current in one coil on another coil, there is also an induced EMF in the first coil ITSELF!

- That is (or should be) completely expected:
  • a coil carrying a changing current, itself experiences the changing flux as well, and thus should itself experience an induced EMF...

- the *self-induced* EMF implies the existence of a *voltage difference* $\Delta V$ between the terminals of the coil ...
  • that opposes the original current change ...

Taking the definition of mutual inductance, we can easily define the **SELF-INDUCTANCE** $L$:

$$\xi_1 = -L \frac{di_1}{dt} = -\Delta V_{\text{coil}}$$  (Note the "symbolic" $-$ sign!)

where $\xi_1$ is the **self-induced** EMF due to the changing current $i_1$...
Calculating Self-Inductance

Self-inductance of a long solenoid (cross section $A$, $N$ windings, length $l$):

$$B = \begin{cases} K_m \mu_0 Ni/l & \text{inside and near the center of the long solenoid} \\ \approx 0 & \text{outside an idealized long solenoid} \end{cases}$$

$$\Rightarrow -E = N \frac{d\Phi_B}{dt} = N \frac{d}{dt} \int_B \mathbf{B} \cdot d\mathbf{A} = N \frac{d}{dt} (BA) = NA \frac{dB}{dt}$$

$$= NA \frac{K_m \mu_0 N}{l} \frac{di}{dt} = L \frac{di}{dt} \Rightarrow L_{\text{solenoid}} = \frac{K_m \mu_0 AN^2}{l} = \frac{\mu AN^2}{l}$$

Self-inductance of a solenoidal toroid (cross section $A$, $N$ windings, average radius $r$):

$$B = \begin{cases} K_m \mu_0 Ni/(2\pi r) & \text{inside the solenoidal toroid} \\ \approx 0 & \text{outside the toroid} \end{cases}$$

$$\Rightarrow -E = N \frac{d\Phi_B}{dt} = N \frac{d}{dt} \int_B \mathbf{B} \cdot d\mathbf{A} = N \frac{d}{dt} (BA) = NA \frac{d\bar{B}}{dt}$$

$$= NA \frac{K_m \mu_0 N}{2\pi r} \frac{di}{dt} = L \frac{di}{dt} \Rightarrow L_{\text{sol. toroid}} = \frac{K_m \mu_0 AN^2}{2\pi r} = \frac{\mu AN^2}{2\pi r}$$