Classical Physics II

PHY132
Lecture 17
Induction
Current-carrying wires exert forces on each other:

Consider the simple case of two parallel long straight wires, carrying currents $I_1$ and $I_2$:

- the first wire’s current $I_1$ generates a magnetic field $B_1$ everywhere, in particular along the location of the second wire...
- the second wire’s current $I_2$ interacts (via the Lorentz force) with the local magnetic field and experiences a force $F_2$

- Calculating:
  \[B_1(\text{at wire 2}) = \frac{\mu_0 I_1}{2\pi d} (\otimes)\]

  \[\Rightarrow F_2 = I_2 l \times B_1 = I_2 l B_1(\uparrow)\]

  \[\Rightarrow \frac{F_2}{l} = I_2 B_1 = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} (\uparrow)\]

- Note the reaction force $F_1 = -F_2$!
- The unit of current A(mperes) is defined in terms of SI units N and m this way, and therefore $\mu_0 = 4\pi \times 10^{-7} \text{N/A}^2$ exactly!
Magnetic Materials

Like dielectric materials in electric fields, materials often have magnetic properties as well, characterized by the relative magnetic permeability $K_m$ multiplying $\mu_0$:

- Thus: $\mu_0 \rightarrow \mu \equiv K_m \mu_0$ in the formulae!

We distinguish three categories:

- **Paramagnetic** materials:
  intrinsic atomic magnetic moments (electron orbital and electron intrinsic spin) align with the external magnetic field; alignment is only approximate as thermal “jitter” disturbs alignment:
  $$K_m \approx 1.0001 - 1.001$$ (small)

- **Diamagnetic** materials:
  some materials without intrinsic magnetic moment can acquire an INDUCED magnetic moment (rotating charge distributions that are pulled apart) which anti-aligns with the external magnetic field:
  $$K_m \approx 0.998 - 0.999$$ (small)

- **Ferromagnetic** materials:
  in materials like Fe, local “domains” of mutually aligned intrinsic spins arise spontaneously. Even a moderate external field will align these domains, which gives an enormous boost to the magnetic field:
  $$K_m \approx 1000 - 10^5$$ (very large)
Ampère’s Law

Consider again Ampere’s law: \[ \oint B \cdot ds = \mu_0 I_{\text{enclosed}} \]
- For a circle around a wire carrying a current \( I \):
- Symmetry tells us that \( B \) cannot have radial components and \( B \) must be tangential to the circle everywhere and constant in magnitude on a given circle of radius \( r \)
- The enclosed current is the current density integrated over any surface enclosed by the circle: \( I_{\text{enclosed}} = \int j \cdot dA \)

\[ \oint B \cdot ds = B2\pi r = \mu_0 I_{\text{enclosed}} = \mu_0 I \implies B = \frac{\mu_0 I}{2\pi r} \]

What if we use the current \( I = dQ/dt \) to charge a parallel-plate capacitor \( C \)?
- The current through a PLANE enclosed by the circle: \( I_{\text{enclosed}} = I \)
- The current through the surface of a “bag” (dashed shape): \( I_{\text{enclosed}} = 0 \neq I \)
- Abandon Ampere’s Law?
Improving Ampere’s Law

The problem is that the LHS of Ampere’s Law depends on the curve (circle’s edge), whereas the RHS depends on the surface enclosed by the circle...

Repairing Ampere’s Law (Maxwell):
- consider what happens between the plates of a charging capacitor: an increasing electric field:

\[ Q(t) = CV(t) = (\varepsilon A/d)(Ed) = \varepsilon AE = \varepsilon \Phi_E \]

- The charging current thus equals:

\[ I = \frac{dQ}{dt} = \varepsilon \frac{d(AE)}{dt} = \varepsilon \frac{d\Phi_E}{dt} \]

- The RHS also exists BETWEEN the plates, and if we ADD this to the wire’s current, Ampere’s Law is saved:

\[ \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \left( I_{\text{enclosed}} + \varepsilon \frac{d\Phi_E}{dt} \right) = \mu_0 \left( I_{\text{enclosed}} + I_D \right) \]

- The last term is the “Maxwell Displacement Current”
Maxwell’s Displacement Current

We can now apply the “new and improved” Ampere’s Law to an surface area that goes between the plates:

- Consider two “Amperian Circles” (red dashed circles):

  For \( r < R \): 
  \[
  \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon \frac{d\Phi_E}{dt} = \mu_0 \varepsilon \frac{dE}{dt} \pi r^2 = \mu_0 \varepsilon \frac{i_C}{\varepsilon A} \pi r^2 \\
  = \mu_0 \frac{i_C}{\pi R^2} \pi r^2 \Rightarrow B(r \leq R) = \frac{\mu_0 i_C}{2\pi} \frac{r}{R^2}
  \]

  For \( r > R \): 
  \[
  \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon \frac{d\Phi_E}{dt} = \mu_0 \varepsilon \frac{dE}{dt} \pi R^2 = \mu_0 \varepsilon \frac{i_C}{\varepsilon A} \pi R^2 \\
  = \mu_0 \frac{i_C}{\pi R^2} \pi R^2 \Rightarrow B(r \geq R) = \frac{\mu_0 i_C}{2\pi r}
  \]

just like the field of a wire of radius \( R \) and a uniform current density!

Note that the charging current in a ERC circuit equals:

\[
  i_C = i_0 e^{-\frac{t}{RC}} = \frac{V}{R} e^{-\frac{t}{RC}}
\]
Varying Magnetic Flux

- The AMAZING conclusion from the above discussion is that (between the plates) a magnetic field is generated by a time-varying electric flux $d\Phi_E/dt$!

- We will now discuss the mirror effect: a time-varying magnetic flux $d\Phi_B/dt$ induces an electric field!

- Magnetic Flux defined: $\Phi_B \equiv \int \mathbf{B} \cdot d\mathbf{A}$
  - Note Gauss' Law for Magnetism: $\oint \mathbf{B} \cdot d\mathbf{A} = 0$

- Consider a conducting wire of length $L$ moving with a velocity $\mathbf{v}$ perpendicular to a uniform magnetic field $\mathbf{B}$:
Faraday’s Induction Law

Recap the definition of Magnetic Flux:
\[ \Phi_B = \int \mathbf{B} \cdot d\mathbf{A} \]
- Note: for a CLOSED surface:
\[ \oint \mathbf{B} \cdot d\mathbf{A} = 0 \]

Gauss' Law for Magnetism; absence of magnetic monopoles

Now consider a \( L \) m long piece of conducting rod moving with velocity \( v \) perpendicular to a uniform and constant magnetic field \( \mathbf{B} \):

the Lorentz force on charge carriers \( q \) in the rod is:
\[
\mathbf{F}_L = q \mathbf{v} \times \mathbf{B}(-\mathbf{k}) = q \mathbf{v} \begin{vmatrix} i & j & k \\ 0 & 0 & -B \\ 0 & 0 & 0 \end{vmatrix} = -j(-qvB) = qvBj
\]

- the \( \mathbf{F}_L \) pushes charges ...
- to accumulate at the ends of the rod
- an EMF is created:
\[
\overline{\mathcal{E}} = \Delta V = \int_{0}^{L} \mathbf{E} \cdot d\mathbf{l} = -\frac{\mathbf{F}_L}{q} L = -vBL
\]
\[
= -BL \frac{dx}{dt} = -B \frac{dA}{dt} = -\frac{d\Phi_B}{dt}
\]
Electromotive Force due to $d\Phi/dt$

Experiments (Faraday et al.) have shown that: 

**Changes in Magnetic Flux** over a surface cause an **Induced EMF** in the curve bounding the surface.

**Faraday's Law:** \[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \]

Thus, a changing magnetic flux can result from:
- a change in field $B$: i.e. $dB/dt \neq 0$
- a change in area $A$: i.e. $dA/dt \neq 0$
- and/or a change in the angle between $\mathbf{B}$ and $\mathbf{A}$: i.e. $d\cos\theta_{A,B}/dt \neq 0$

We'll consider examples of each of these in the following...
Changing Magnetic Field → EMF

Consider a loop of \( N \) windings and area \( A \) immersed in a uniform magnetic field \( B \), making an angle of 30° with the normal to the plane of the loop.

Assume the field is changing over time as: \( B = B_0 \, e^{-t/\tau} \) (e.g. due to a charging capacitor) with \( \tau \) some characteristic time constant:

- Note the field is DECREASING with time:
  \[
  \frac{dB}{dt} = -(B_0/\tau) \, e^{-t/\tau} \leq 0
  \]
  \[
  \text{and} \quad E = -d\Phi_B/dt = -[B_0NA\cos(30^\circ)/\tau] \, e^{-t/\tau}
  \]
- At \( t=0 \):
  \[
  E_0 = -d\Phi_B/dt|_{t=0} = -B_0NA\cos(30^\circ)/\tau
  \]
- At \( t=\infty \):
  \[
  E_0 = -d\Phi_B/dt|_{t=\infty} = 0
  \]

The meaning of the negative sign in \( E = -d\Phi_B/dt \):

- The induced EMF is such that it opposes the change in flux;
  - i.e., the resulting current in the loop will be \( I_{\text{induced}} = E/R \),
  - and the induced current will flow in such a direction that it opposes the change in flux (a decrease for this example),

- i.e. here it will be such that \( B_{\text{induced}} \) strengthens the instantaneous field which is decreasing with time...

This is Lenz’ Law
Using Lenz’ Law, determine the direction of the induced current in circuit 2 \((a \rightarrow b \text{ or } b \rightarrow a)\) when:

- with switch \(S\) closed, coil 2 is moved CLOSER to coil 1:
  coil 2 moves into a stronger B field (B to right) \(\Rightarrow\) flux in coil 2 increases.
  Lenz’ Law: current induced in 2 opposes this \(\Rightarrow\) induced field to the left; i.e. induced current in 2: \(a \rightarrow b\)

- keeping the switch closed, resistance \(R\) is reduced:
  Field (to right) from coil 1 becomes stronger \(\Rightarrow\) flux in coil 2 increases.
  Lenz’ Law: current induced in 2 opposes this \(\Rightarrow\) induced field to the left; i.e. induced current in 2: \(a \rightarrow b\)

- after having been closed, switch \(S\) is now opened:
  when switch opens: B field (B to right) suddenly decreases \(\Rightarrow\) flux in 2 decreases.
  Lenz’ Law: current induced in 2 opposes this \(\Rightarrow\) induced field to the right; i.e. induced current in 2: \(b \rightarrow a\)