Classical Physics II

PHY132
Lecture 15
Ampère’s Law
Magnetic Field from a Wire Current

We now return to the GENERATION of the magnetic field \( \mathbf{B} \) by a current...

The simplest case is the generation of a \( \mathbf{B} \) field by a infinitely long straight wire carrying a constant current \( I \)

- **Biot–Savart Law:**

\[
d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I dl \times \mathbf{r}}{r^2}
\]

- gives the contribution \( d\mathbf{B} \) to the field due to the current \( I \) in a (infinitesimally) short piece of the wire \( dl \) in point \( r \) away from \( dl \).
Magnetic Field from a Long Wire

Calculate the Magnetic Field at point \( P \), at a distance \( x \) from a long wire carrying a current \( I \):
- Consider a segment \( dy \) of the wire, at position \( y \) above the origin...
- Using the right-hand-rule, all contributions to the \( B \)-field are in the same \(-z\) direction!

\[
\begin{align*}
\mathbf{dB}_z &= \frac{\mu_0 I}{4\pi} \frac{y \mathbf{\hat{y}} \times \mathbf{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{y \sin \alpha'}{r^2} (-\mathbf{k}) \\
B(x) &= \frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{dy}{r^2} \cos \alpha
\end{align*}
\]

\[
B(x) = B(x)(-\mathbf{k}) = \int \mathbf{dB}_z(x)
\]

Choose \( \alpha \) as the integration variable; then:

\[
y = x \tan \alpha \quad \Rightarrow \quad dy = \frac{x}{\cos^2 \alpha} \cos \alpha
\]

\[
r = \frac{x}{\cos \alpha} \quad \Rightarrow \quad \frac{1}{r^2} = \frac{\cos^2 \alpha}{x^2}
\]

Thus:

\[
B(x) = \frac{\mu_0 I}{4\pi x} \int_{-a}^{a} \cos \alpha \, d\alpha = \frac{\mu_0 I}{4\pi x} \int_{-a/\sqrt{a^2+x^2}}^{a/\sqrt{a^2+x^2}} d(\sin \alpha)
\]

\[
B(x) = \frac{\mu_0 I}{4\pi x} \frac{2a}{\sqrt{a^2+x^2}} (-\mathbf{k}) = \frac{\mu_0 I}{2\pi x} \frac{a/x}{\sqrt{1+a^2/x^2}} (-\mathbf{k})
\]

\[
\lim_{a \gg x} B(x) = \frac{\mu_0 I}{2\pi x} \frac{a/x}{\sqrt{a^2/x^2}} = \frac{\mu_0 I}{2\pi x}
\]

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Symmetry!

The symmetry of the long straight wire problem, allows the use of a powerful tool, SYMMETRY, to constrain the possible shapes of the magnetic field:

1. **The wire has a rotational symmetry**: rotating the wire around itself doesn’t change the current (direction and magnitude), and thus the \( \mathbf{B} \)-field must be equal to itself as well under such rotations...

2. **The wire has translational symmetry**: moving the (infinitely long straight) wire in any direction along its length doesn’t change the current, and thus the \( \mathbf{B} \)-field must have the same symmetry...

3. Finally, changing the direction of the wire current to the opposite, i.e. a “flip”, must flip the direction of the \( \mathbf{B} \)-field as well...
   - This excludes radial field lines!

Consequence: the \( \mathbf{B} \)-field can ONLY form circles around the wire:

- Any **radial** \( \mathbf{B} \)-field component is forbidden by 3)
- Any \( \mathbf{B} \)-field component **parallel** to the wire is forbidden by 2) and 3)
Magnetic Field from Wire Loop

Consider a circular wire loop of radius $R$, carrying a current $I$; $O$ is the center:

- The transverse components $dB_y$ cancel in the sum, while the $dB_x$ add:
  - because of the SYMMETRY: rotating the ring around $O$ should NOT change anything $\Rightarrow$ B-field MUST be parallel to the $x$-axis!

- The $x$-components add:

$$B(x) = \frac{\mu_0}{4\pi} \int_0^{2\pi R} \frac{Is}{r^2} \cos \alpha' \, ds = \frac{\mu_0}{4\pi} \int_0^{2\pi R} \frac{Is}{r^2} \sin \alpha = \frac{\mu_0}{4\pi} \frac{I}{R} \int_0^{2\pi R} ds$$

$$B(x) = \frac{\mu_0 I}{2} \frac{R^2}{r^3} \mathbf{i} = \frac{\mu_0 I}{2} \frac{R^2}{\left(R^2 + x^2\right)^{3/2}} \mathbf{i}$$

- Check:
  for $x \gg R$ we obtain:

$$\lim_{x \gg R} B(x) = \frac{\mu_0 I}{2} \lim_{x \gg R} \frac{R^2}{\left(R^2 + x^2\right)^{3/2}} = \frac{\mu_0 I}{2} \frac{R^2}{x^3}$$

$$B(x=0) = \frac{\mu_0 NI}{2R} \mathbf{i}$$

e.g.: $N=100$, $I=5$ A, $R=5$ cm: $B = \frac{2\pi \times 10^{-7} \times 100 \times 5.0}{0.10} = 3.1 \times 10^{-3}$ T
Gauss’ Law for Magnetism

We saw before, that the \( \mathbf{B} \) field lines for a long wire carrying a current \( I \) are circles centered on the wire.

- This is more generally true: the \( \mathbf{B} \)-field lines always form CLOSED CURVES (non-intersecting)
- This is VERY DIFFERENT from the electric filed lines, which start/end on charges (sources)

This is also equivalent to the statement that there exist NO MAGNETIC MONOPOLES...

- there is no experimental evidence for magnetic charges (magnetic “monopoles”)

Hence, Gauss’ Law for Magnetism:

\[
\oint \mathbf{B} \cdot d\mathbf{A} = 0
\]

\( \oint \) closed surface

cfr. Gauss’ Law for the Electric Field:

\[
\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}
\]

\( \oint \) closed surface
Ampere’s Law

the Biot-Savart Law (infinitesimally small current wire segment, or speeding charge) has a $1/r^2$ form,
- it leads to a $1/r$ form for the field of an infinitely long wire,
- and to a constant field for a infinitely large “sheet” of current-carrying wires...

This leads to a “geometric” interpretation of the field: field lines!
The magnetic field lines always form closed loops
- because of Gauss’ Law for magnetism, or the absence of magnetic monopoles ...

Above two observations results in AMPERE’S LAW, which (like Gauss’ Law for Electrostatics) is very useful for finding the B-field in symmetric situations.
- Ampere’s Law is derivable as an (line) integral form of the Biot-Savart Law...

Ampere’s Law:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enclosed}}$$

Ampere’s Law

- where $I_{\text{enclosed}}$ is a current flowing through the surface enclosed by the curve, more specifically:

$$I_{\text{enclosed}} = \int_{\text{surface area bounded by the closed curve}} \mathbf{j} \cdot dA$$
Applying Ampere’s Law

Ampere’s Law is utilized by computer programs for field calculations.

For our purposes it is mostly useful in symmetric situations

- e.g. a long straight wire with current $I$ (into the paper):
- the B-field lines are circles around the wire:
- Ampere’s Law applied to the dashed circle:

$$\oint_{\text{circle radius } r} \mathbf{B} \cdot d\mathbf{s} = B \int_0^{2\pi r} ds = B2\pi r$$

$$= \mu_0 I_{\text{enclosed}} = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

- as we found before!
We'll now use Ampere's Law for other symmetric situations:

The field of a very long solenoid:

- **Ampere's Law** applied to the dashed rectangle:
  - the $\mathbf{B} \cdot d\mathbf{l}$ vanishes for the left & right sides because $\mathbf{B}$ is (nearly) perpendicular to these sides
  - the $\mathbf{B} \cdot d\mathbf{l}$ vanishes for the top side because $\mathbf{B}$ is (nearly) zero there...

- Thus:
  $$\int_{\text{rectangle}} \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enclosed}}$$
  $$= B_{\text{sol}} l = \mu_0 N I \quad \Rightarrow \quad B = \frac{\mu_0 N I}{l}$$

![Rectangular Amperian Loop](image)

N windings enclosed by the rectangle
Magnetic Field from a Sheet of Current

The field of a wide and long “sheet” of current:

- current $I$ and width $L$
- Ampere’s Law applied to the dashed rectangle:
  - the $\mathbf{B} \cdot d\mathbf{l}$ vanishes for the left & right sides because $\mathbf{B}$ is (nearly) perpendicular to these sides
  - the $\mathbf{B} \cdot d\mathbf{l}$ does not vanishes for the top and bottom sides and $\mathbf{B}$ is constant there...
- Thus:

$$
\oint_{\text{rectangle}} \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enclosed}}
$$

$$
= 2B_{\text{sol}} l = \mu_0 \frac{I}{L} l
\Rightarrow
B = \frac{\mu_0 I}{2L}
$$

ideal current sheet
A solenoidal toroid is a solenoid bent around into itself.

The solenoidal toroid shown has an inner radius \(a\), outer radius \(b\), and \(N\) windings and current \(I\) ...

- The Amperian curve is the dashed circle which is located within the windings: \(a < r < b\)
- Symmetry tells us that the B-field lines are concentric circles...
- The field outside the solenoidal toroid is (nearly) zero...

- Ampere’s Law: 
  \[
  \oint_{\text{circle, } r} \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enclosed}}
  \]
  \[
  = B_{\text{tor.sol}} 2\pi r = \mu_0 NI \implies B = \frac{\mu_0 NI}{2\pi r}_{\text{solenoidal toroid}}
  \]