Classical Physics II

PHY132
Lecture 8

Capacitance, Dielectrics, and Stored Energy
Recap

• Electric Field $E$
  - Generated by charged “sources”,
  - VECTOR field that pervades all space: $E = (E_x(r), E_y(r), E_z(r))$
  - a “test charge” interacts with $E$ and experiences a force $F = qE$.
  - may be represented by “field lines”, giving direction and strength

• Gauss’ Law: $\Phi_E = \oint_{\text{closed surface}} E \cdot dA = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$
  - Coulomb’s Law for point-like charges: $F_E = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{r^2} \hat{r}$

• Electric Potential:
  - potential energy change: $\Delta U_E = -\text{Work done by E}$
  - electric potential: $V = U_E/q_{\text{test}}$
    $\Delta V = V(r_2) - V(r_1) = -\int_{r_1}^{r_2} E \cdot ds$ and: $E = -\frac{dV}{dr}$

• Capacitance:
  $C \equiv \frac{Q(r)}{V(r)}$ or: $Q = CV$
the Potential energy of the arrangement of three charges below equals ...

A. $3kq^2/L^2$
B. $2kq^2/L$
C. $3kq^2/L$
D. not simply calculable without calculator ...
A charged set of concentric (conducting) spheres of radii \(a\) and \(b > a\), with charges \(+Q\) and \(-Q\) respectively:

\[
E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \quad \text{for } a \leq r \leq b; \text{ elsewhere } E = 0!
\]

\[
\Delta V(b \rightarrow a) = V(a) - V(b) = -W(b \rightarrow a)/q_{\text{test}} = W(a \rightarrow b)/q_{\text{test}}
\]

\[
= \int_a^b \frac{1}{4\pi\varepsilon_0} \frac{Qq_{\text{test}}}{r^2} \, dr / q_{\text{test}} = \frac{Q}{4\pi\varepsilon_0} \int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0} \left[ \frac{-1}{b} - \frac{-1}{a} \right]
\]

\[
= \frac{Q}{4\pi\varepsilon_0} \left[ \frac{1}{a} - \frac{1}{b} \right] = \frac{Q}{4\pi\varepsilon_0} \frac{b - a}{ab}
\]

\[
\Rightarrow C_{\text{concentric spheres of radii } a,b>a} = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{b-a} \frac{ab}{b-a}
\]

Note: \(C\) depends only on geometrical quantities (i.e. sizes)!
the Capacitance of a Cylinder ...

A long charged cylindrical conductor with charge-per-unit length \( Q/L \), and radius \( R \):

\[
E(r) = \frac{1}{2\pi\varepsilon_0} \frac{Q}{L} \hat{r}
\]

\[
\Delta V(\mathbf{r}_0 \rightarrow \mathbf{r}) = V(\mathbf{r}) - V(\mathbf{r}_0) = -W(\mathbf{r}_0 \rightarrow \mathbf{r})/q_{test} = W(\mathbf{r} \rightarrow \mathbf{r}_0)/q_{test}
\]

\[
= \int_{r_0}^{r} \frac{1}{2\pi\varepsilon_0} \frac{Qq_{test}}{L} \hat{r}_n \cdot d\mathbf{l} / q_{test} = \frac{Q}{2\pi\varepsilon_0 L} \int_{r_0}^{r} \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L} \ln \frac{r}{r_0}
\]

\[
= \frac{1}{2\pi\varepsilon_0} \frac{Q}{L} \ln \frac{R}{r} - \frac{1}{2\pi\varepsilon_0} \frac{Q}{L} \ln \frac{R}{r_0}
\]

Note: the only obvious choice for a radial distance at which we can set \( V=0 \), is \( r=R! \)

\[
E(r) = \frac{1}{2\pi\varepsilon_0} \frac{Q}{L} \hat{r} \quad \text{(for } a \leq r \leq b)\]

However: TWO concentric (co-axial) cylinders of radii \( a \) and \( b>a \):

\[
\Delta V(b \rightarrow a) = V(a) - V(b) = -W(b \rightarrow a)/q_{test} = W(a \rightarrow b)/q_{test}
\]

\[
= \int_{a}^{b} \frac{1}{2\pi\varepsilon_0} \frac{Qq_{test}}{L} dr / q_{test} = \frac{Q}{2\pi\varepsilon_0 L} \int_{a}^{b} \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L} \ln \frac{b}{a}
\]

\[
\Rightarrow C_{\text{co-axial cyl's; radii } a,b>a} = \frac{Q}{\Delta V} = 2\pi\varepsilon_0 \frac{L}{\ln (b/a)} \Rightarrow \frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln (b/a)}
\]
Parallel Plate Capacitor

For TWO parallel planes with opposite charges $+Q$ and $-Q$, each with area $A$, and separated by distance $d << \text{size of the planes}$, we find:

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{\varepsilon_0} \hat{x} = \frac{Q}{\varepsilon_0 A} \hat{x} \quad \text{(in between the planes!)}$$

$$\Delta V(d \to x) = V(x) - V(d) = -W(d \to x)/q_{\text{test}} = W(x \to d)/q_{\text{test}}$$

$$= \int \limits_x^d \frac{Qq_{\text{test}}}{\varepsilon_0 A} \hat{x} \cdot d\mathbf{x}/q_{\text{test}} = \frac{Q}{\varepsilon_0 A} \int \limits_x^d dx = \frac{Q}{\varepsilon_0 A} [d - x]$$

$$\Rightarrow V(x) = \frac{Q}{\varepsilon_0 A} (d - x)$$

$$\Rightarrow V(x=0) = \frac{Q}{Ed} = \frac{A}{\varepsilon_0 d}$$

Note: again $C$ depends only on geometrical quantities (i.e. sizes)!

- Example: two parallel plates of area $A = 930 \text{ cm}^2 (1 \text{ sqft})$ and separated by $d = 0.80 \text{ mm} (~1/32")$

  - Capacitance for a single plane, $d \to \infty$, is zero.
  - Capacitance: $C = \varepsilon_0 A/d = 8.85 \times 10^{-12} \text{ F/m} \times 930 \times 10^{-4} \text{ m}^2/0.80 \times 10^{-3} \text{ m} = 1.0 \times 10^{-9} \text{ F} = 1.0 \text{ nF}$
  - Charge for $\Delta V = 100 \text{ V}$: $Q = CV = 1.0 \text{ nF} \times 100 \text{ V} = 100 \text{ nC} = 0.10 \mu\text{C}$
the Capacitance of a Parallel-Plate capacitor …

A. increases when the distance between the plates increases …

B. increases when the charge on the plates increases …

C. increases when the area of the plates increases …

D. all of the above …
Capacitors are thus charge storage devices:
• in real life they are formed by a set of parallel conductors, with connector tabs and wire leads
• and with an insulator in between (air, plastic, oil, …)

- symbol: \[ C \]
- units: \[ [C] = [Q/V] = \text{Coulomb}/V(olt) = C^2/J \equiv F(\text{arad}) \]

Systems of Capacitors:
• PARALLEL connection:

\[
\begin{align*}
C_{\text{total},||} &= \frac{Q}{V_b - V_a} = \frac{Q}{\Delta V_{ba}} = \frac{Q_1 + Q_2}{\Delta V_{ba}} = \frac{Q_1}{\Delta V_{ba}} + \frac{Q_2}{\Delta V_{ba}} = C_1 + C_2
\end{align*}
\]

• SERIES connection:

\[
\begin{align*}
\Delta V_{ba} &= \frac{Q}{C_{\text{total}}} = V_b - V_a = V_b - V_c + V_c - V_a \\
&= \Delta V_{bc} - \Delta V_{ca} = \frac{Q}{C_1} + \frac{Q}{C_2} \quad \Rightarrow \quad \frac{1}{C_{\text{total}, \text{series}}} = \frac{1}{C_1} + \frac{1}{C_2}
\end{align*}
\]
I have three 3 \( \mu \text{F} \) capacitors available ...

A. when I connect them in parallel they make 1 \( \mu \text{F} \)
B. when I connect them in parallel they make 9 \( \mu \text{F} \)
C. when I connect them in series they make 1 \( \mu \text{F} \)
D. when I connect them in series they make 9 \( \mu \text{F} \)
E. A and D
F. B and C
Capacitors store charge and thus energy: connecting the leads together causes a current flow from $+Q$ to $-Q$! The stored energy can be calculated as the result of the process of starting with a set of uncharged conductors, and transferring small increments of charge $dq$ from one conductor to the other, until the charges on the conductors are $+Q$ and $-Q$ respectively, and the potential difference is $\Delta V$:

- assume that at some point in the process the potential is $q/C$; then the (external!) work done in the next transfer of $dq$ equals:

$$dW_{ext} = \frac{q}{C} dq \quad \Rightarrow W_{ext} = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{1}{2} CV^2 = Q\left(\frac{1}{2} V\right) = U$$

where $U$ is the STORED ELECTRICAL ENERGY

- the Energy Density $u$ is then:

$$u = \frac{U}{\text{Volume}} = \frac{1}{2} CV^2 = \frac{1}{2} \varepsilon_0 \left(\frac{A}{d}\right) V^2 = \frac{1}{2} \varepsilon_0 \frac{V^2}{d^2} = \frac{1}{2} \varepsilon_0 E^2$$

average voltage over the charging process…
Dielectrics

**Dielectrics** are insulating materials that become electrically polarized in an external electric field:
- the overall-neutral molecules are (or become induced) dipoles,
  - that anti-align themselves with an external field (opposing it).
- this effectively *reduces* the local electric field,
  - and thus reduces the *potential difference* across the dielectric relative to the non-polarized case!
- typically, assuming the charge $Q$ on the capacitor remains constant (isolated charged capacitor): $E_{\text{in dielectric}} = E_{\text{in vacuum}}/K$

thus, for a parallel-plate capacitor:

$$
\Rightarrow V' = E' d = \frac{E_{\text{vac}}}{K} d = \frac{V_{\text{vac}}}{K} \quad \Rightarrow C' \equiv \frac{Q}{V'} = K \frac{Q}{V_{\text{vac}}} = KC_{\text{vac}} = K \varepsilon_0 \frac{A}{d} \equiv \varepsilon \frac{A}{d}
$$
- Thus, inserting a dielectric in between the conductors that form a capacitor, *increases* the capacitance by $K$
  - $K$ is a material-dependent “constant” (depends on temperature, frequency, etc.)
which of the following is true?
When a dielectric with $K=2$ is inserted between the plates of a parallel-plate capacitor (fully filling the gap), ...

- A. the capacitance doubles ...
- B. the charge on the plates doubles ...
- C. the voltage difference between the plates doubles ...
- D. the voltage difference between the plates is halved ...
- E. A and B and C
- F. A and B and D
- G. A and D
Example

two parallel plates of area $A = 93 \text{ cm}^2 \ (0.1 \text{ sqft})$ are separated by dielectric material $d = 25 \ \mu \text{m} \ (1/1000")$ with dielectric constant $K = 5.0$:

- **Capacitance:**
  \[ C = K \varepsilon_0 A/d = 5.0 \times 8.85 \times 10^{-12} \text{F/m} \times 93 \times 10^{-4} \text{m}^2 / 25 \times 10^{-6} \text{m} = 16.5 \times 10^{-9} \text{F} = 16.5 \text{ nF} \]

- **Charge for $\Delta V = 100 \text{ V}$:**
  \[ Q = CV = 16.5 \text{ nF} \times 100 \text{ V} = 1650 \text{ nC} = 1.65 \mu \text{C} \]

- **Electric Field:**
  \[ \Delta V = Ed \rightarrow E = V/d = 100 \text{ V} / 25 \times 10^{-6} \text{m} = 4.0 \text{ MV/m} \]

- **Stored energy for $\Delta V = 100 \text{ V}$:**
  \[ U = \frac{1}{2} CV^2 = \frac{1}{2} \times 16.5 \text{ nF} \times 100^2 \text{ (J/C)}^2 = 8.2 \times 10^{-9} \text{ J} \]
  (i.e. not a big amount!)

- **Alternatively: Stored energy:**
  \[ U = \text{Volume} \times u = Ad \times \frac{1}{2} K \varepsilon_0 E^2 \]
  \[ = 93 \times 10^{-4} \text{m}^2 \times 25 \times 10^{-6} \text{m} \times \frac{1}{2} 8.85 \times 10^{-12} \text{F/m} \times (4 \times 10^6 \text{V/m})^2 = 8.2 \times 10^{-5} \text{J} \]
Current – Flow of Charges

In the preceding notes, we have claimed that the electric field \( E \) inside a conductor must always be zero
- if not, the mobile charges will re-arrange themselves such as to cancel the electric field \( E \):

\[
E = 0
\]

- note that only a small percentage of the charges (electrons here) need to rearrange themselves in order to cancel even a strong external electric field...

⇒ If a net electric field exists inside a conductor (i.e. a Voltage gradient: \( E_x = -dV/dx \), ...) an electric current must flow:
- Current \( I \equiv dQ/dt \), units: A(mpere) = C/s
- this current dies out very quickly, unless the build-up of charges at the ends is prevented by continuously delivering (source-ing) and removing (sink-ing) them at the ends!
- A well-charged capacitor, or a BATTERY may provide such...
Resistance

Let's assume we have somehow arranged (e.g. using a battery as source and sink of charges) for a permanent uniform electric field \( \mathbf{E} \) inside a non-ideal conducting wire of cross section \( A \) and length \( L \):

then there must exist a potential difference (voltage difference) between the ends of the wire: \( V = \int_L^0 \mathbf{E} \cdot d\mathbf{l} = EL \)

- An electric field acting on a mobile charge \( q \) exerts a force \( \mathbf{F} = q \mathbf{E} \), thus we expect the charge carriers to be accelerated (if no friction!)
  
  • Typically, the carriers bounce around inside the conductor \( <K> = \frac{3}{2}kT \) at high speeds (electrons in metal \( v_{\text{rms}} \sim 10^6 \text{ m/s} \)).
  
  • In-between collisions they get accelerated in the direction of the electric force, but in the next collision they loose all memory of direction... The net result of the electric force is an overall “drift” of the electron “cloud” inside the conductor in the direction of the electric force...

  • Typically \( v_d \) is very small!
  
  • Collisions dissipate the energy \( \implies \) wire heats up

- **EMPIRICALLY**: \( v_d (= v_{\text{drift}}) \propto E = \Delta V/L \)
Resistance

The resulting current through the conducting wire of $\varnothing A$ is

$$I = \frac{dQ}{dt} = q \, n_q \,(Adx) /dt = qn_q \, Av_d,$$

- where $n_q$ is the density of charge carriers in the conductor (#/m$^3$),
- and $q$ the charge of a single carrier; we assume that all carriers are of the same type (e.g. electrons).

Because $v_d$ is proportional to $E$:

$$I = qn_q \, Av_d \propto qn_q \, AV/L \rightarrow E=V/L = \rho \, I/A \equiv \rho j$$

where $j$ is the “current density” in the wire.

- The proportionality constant $\rho$ is dependent on material, temperature, and typically not even constant with voltage…
- the material “constant” $\rho$ is called “resistivity”
- the total resistance $R$ of the wire is defined as:

$$R = V/I = (V/L)/(I/A) \times A/L = \rho A/L$$

- Units: $[j] = A/m^2 = C/m^2/s$,
- Resistance: $[R] = V/A \equiv \text{Ohm} = \Omega$
- Resistivity: $[\rho] = (V/m)/(A/m^2) = Vm/A \equiv \text{Ohm} \cdot \text{m} = \Omega \text{m}$

Ohm’s “Law”:

$$V = IR, \quad \text{or:} \quad V/L = \rho \, I/A$$
Drift Speed

A Cu wire carries a current of 1 A, and has a diameter of 1 mm²; then:

\[ n_e = \frac{8.9 \times 10^6 \text{ g/m}^3/(63.5 \text{ g/mol/}6.0 \times 10^{23} \text{ atom/mol}) \times \sim 1 \text{ e}^-/\text{atom}}{} \]
\[ = 8.4 \times 10^{28} \text{ e}^-/\text{m}^3 \]

The current \( I \) is related to the average drift speed \( v_d \):

\[ I = en_e A v_d \]

\[ v_d = \frac{I}{(en_e A)} = \frac{1}{(1.6 \times 10^{-19} \times 8 \times 10^{28} \times 1 \times 10^{-6})} = 8 \times 10^{-5} \text{ m/s} \]

\[ \text{~i.e. if the wire is 1 m long, it takes an electron (on the average) 1.25 \times 10^4 \text{ s (3.5 hr)} to travel the full length of the wire ...} \]

- (now that is not quite correct: an electron’s average thermal speed is extremely high, so it moves all over the place very rapidly!)
**Resistors**

**Typical metals** have low resistivity (high conductivity):

\[ \rho \approx 1-100 \times 10^{-8} \, \Omega \text{m} = 0.01-1 \times 10^{-2} \, \Omega \text{mm}^2/\text{m} \]

**Typical insulators** have high resistivity:

\[ \rho \geq 10^8 \, \Omega \text{m} \]

In electronics, resistors are either metal-wire/film or carbon resistors, with color coded values...

**Common resistor values:** \( R = 0.1 \Omega - 100 \, \text{M}\Omega \) (±1%, 5%, 10%)

- **Symbols:**
  
  (a thin wire wrapped on an insulating rod)

- **Systems of resistors:**
  - **Series:** \( R_{\text{tot}} = R_1 + R_2 \)
  
  - **Parallel:**
    
    \( I_{\text{tot}} = I_1 + I_2 \implies V_{ab}/R_{\text{tot}} = V_{ab}/R_1 + V_{ab}/R_2 \)
    
    \[ \implies 1/R_{\text{tot}} = 1/R_1 + 1/R_2 \]

    (as for Caps in **SERIES**!)
RC Circuit

Connecting a charged capacitor to a resistor gets current to flow, which slowly decreases until the capacitor is fully drained:

After switch $S$ is closed ($t=0$):

$$V(t) = \frac{Q(t)}{C} = IR = -R \frac{dQ}{dt} \Rightarrow \frac{dQ}{dt} = -\frac{1}{RC} Q(t) \Rightarrow Q(t) = Q_0 e^{-\frac{t}{RC}}$$

And equivalently:

$$V(t) = \frac{Q(t)}{C} = \frac{Q_0}{C} e^{-\frac{t}{RC}} = V_0 e^{-\frac{t}{RC}}$$

$$I(t) = \frac{V(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{RC}} = I_0 e^{-\frac{t}{RC}}$$

Note the CHARACTERISTIC TIME $RC$:
- after $t=RC$ charge, current, and voltage are down to $1/e$, after $t=2RC$, they are down to $1/e^2$, etc.

Note the $-$sign: $V(t)$ is positive; $Q(t)$ diminishes with time, so the slope $dQ/dt < 0$!